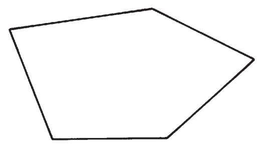
C H A P T E R

THIRTEEN



Tiling with Convex Polygons

> "Many of the brightly coloured, tile-covered walls and floors of the Alhambra in Spain show us that the Moors were masters in the art of filling a plane with similar interlocking figures, bordering each other without gaps. What a pity that their religion forbade them to make images!"
>
> —M. C. Escher

Imagine that you have an infinite supply of jigsaw puzzle pieces, all identical. If it is possible to fit them together without gaps or overlaps to cover the entire plane, the piece is said to tile the plane, and the resulting pattern is called a tessellation. From the most ancient times such tessellations have been used throughout the world for floor and wall coverings and as patterns for furniture, rugs, tapestries, quilts, clothing, and other objects. The Dutch artist M.

C. Escher amused himself by tessellating the plane with intricate shapes that resemble birds, fish, animals, and other living creatures (*see* Figure 75).

A tile that tessellates obviously can have an infinite variety of shapes, but by imposing severe restrictions on the shape, the task of classifying and enumerating tessellations is reduced to something manageable. Geometers have been particularly interested in polygonal tiles, of which even the simplest present formidable problems. In this chapter we are concerned only with the task of finding all convex polygons that tile the plane. It is a task that was not completed until 1967, when Richard Brandon Kershner, assistant director of the Applied Physics Laboratory of Johns Hopkins University, found three pentagonal tilers that had been missed by all predecessors who had worked on the problem.

Let us begin by asking how many of the regular polygons tile the plane. As the ancient Greeks knew and proved, there are just three: the equilateral triangle, the square, and the regular hexagon. The hexagonal tiling, so familiar to bees and users of bathrooms, is a fixed pattern (*see* Figure 76). The patterns formed by equilateral triangles or by squares can be infinitely varied by sliding rows of triangles or squares along lattice lines.

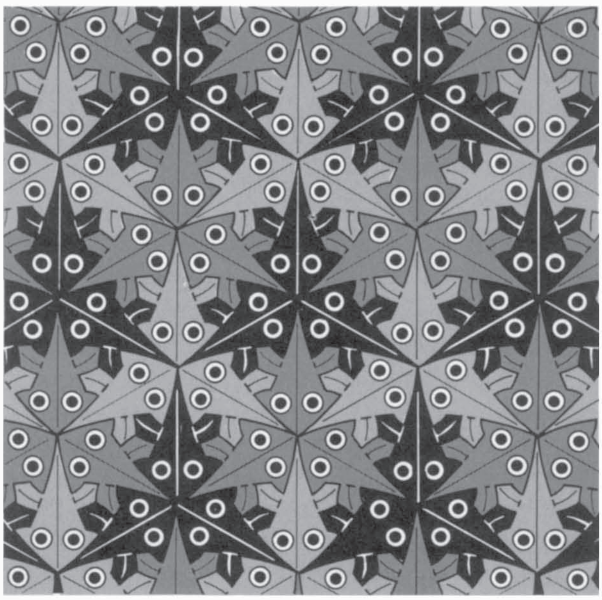


Figure 75 Tessellation by M. C. Escher

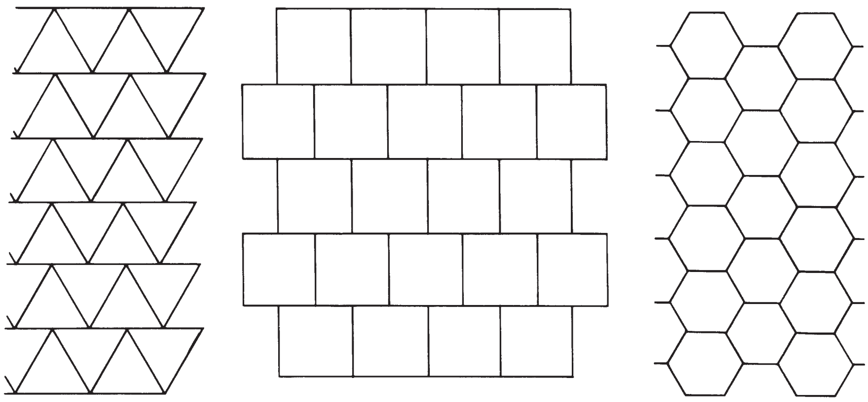


Figure 76 The three regular polygons that tile the plane

If we remove the restriction that a convex polygon must be regular, the tiling problem grows in interest. It has been proved that no convex polygon of more than six sides can tile the plane. Thus we need to investigate only polygons of three, four, five, and six sides.

The triangle is easy. Any triangle tiles the plane. Simply fit two identical triangles together, with the corresponding edges coinciding as shown in Figure 77, and you create a parallelogram. Replicas of any parallelogram obviously will go side by side to make an endless strip with parallel sides, and the strips, in turn, go side by side to fill the plane.

The quadrilateral is almost as easy, although much more surprising. Any quadrilateral tiles the plane! As before, take a pair of identical quadrilaterals, one inverted with respect to the other, join the corresponding edges and you create a hexagon (*see* Figure 78). Each edge of the hexagon is necessarily equal to and parallel to its opposite edge. Such a hexagon, by a simple translation operation (altering its position on the plane without changing its orientation), will form a tiling pattern. The quadrilateral need not be convex. Exactly the same technique creates a tiling pattern for any nonconvex quadrilateral.

The case of the hexagon was settled in 1918 by K. Reinhardt in his doctoral thesis at the University of Frankfurt. He showed that any tessellating convex hexagon belongs to one of three classes. Kershner, in a 1969 article, "On Paving the Plane," explains the three types as follows.

Label the sides and angles of a hexagon as shown in Figure 79. A convex hexagon will tile the plane if and only if it belongs to one or more of the following classes:

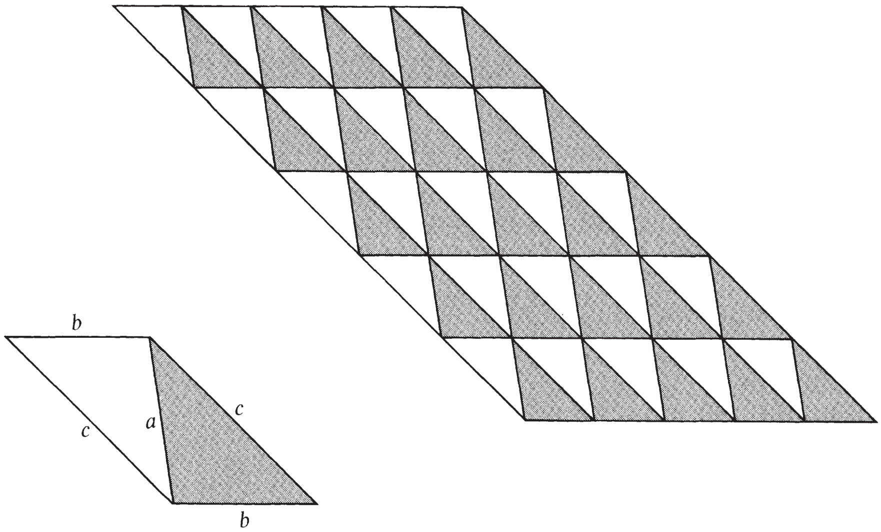


Figure 77 Any triangle tiles the plane

1. $A + B + C = 360°$,
   and $a = d$.
2. $A + B + D = 360°$,
   and $a = d$, $c = e$.
3. $A = C = E = 120°$,
   and $a = b$, $c = d$, $e = f$.

The illustration gives an example of each type of convex hexagon tiler and a portion of its tiling patterns. The gray lines outline a "fundamental region" that tiles the plane by translation. Note that Type 2 requires reflection if the hexagon is asymmetric.

In similar fashion the tessellating convex pentagons can be classified in eight ways. Five were found by Reinhardt. Kershner describes them by labeling the pentagon as shown in Figure 80. A convex pentagon paves the plane if it belongs to one or more of the following classes:

1. $A + B + C = 360°$.
2. $A + B + D = 360°$,
   and $a = d$.
3. $A = C = D = 120°$,
   and $a = b$, $d = c + e$.

4. $A = C = 90°$,
   and $a = b$, $c = d$.
5. $A = 60°$, $C = 120°$,
   and $a = b$, $c = d$.

Examples of each type and its tiling pattern are reproduced in the illustration with gray lines outlining fundamental regions. Only Type 2 requires reflection.

"At this point," writes Kershner, "either [Reinhardt's] technique or his fortitude failed him, and he closed the thesis with the statement that in principle it ought to be possible to complete the consideration of pentagons along the lines of his considerations up to that point, but it would be very tedious and there was always the possibility that no further types would emerge Indeed, it is quite clear that Reinhardt and everyone else in the field thought that the Reinhardt pentagon list was probably complete.

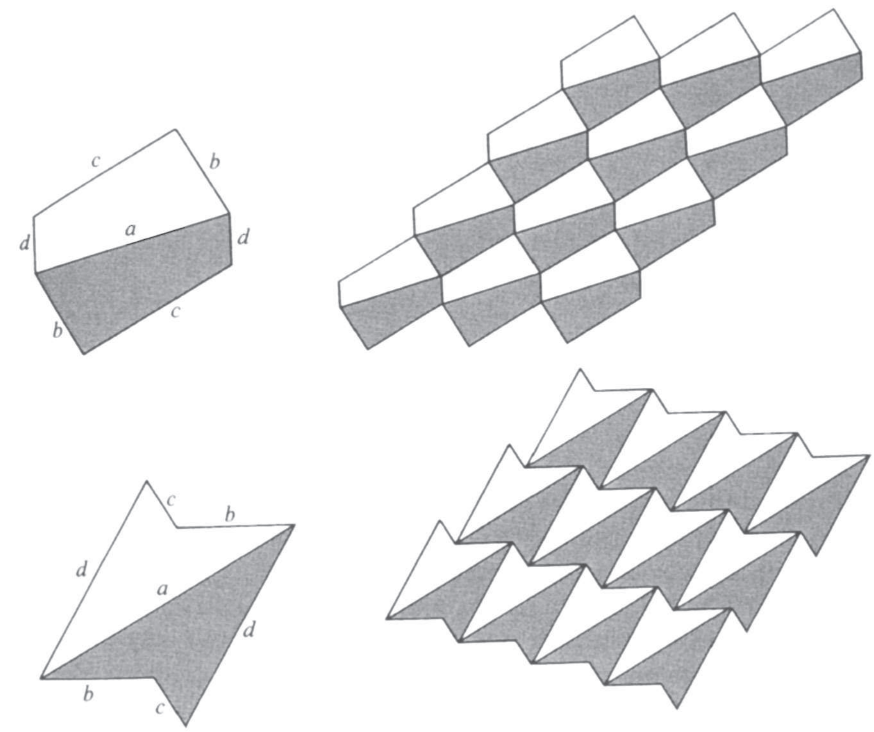


Figure 78 Any quadrilateral tiles the plane

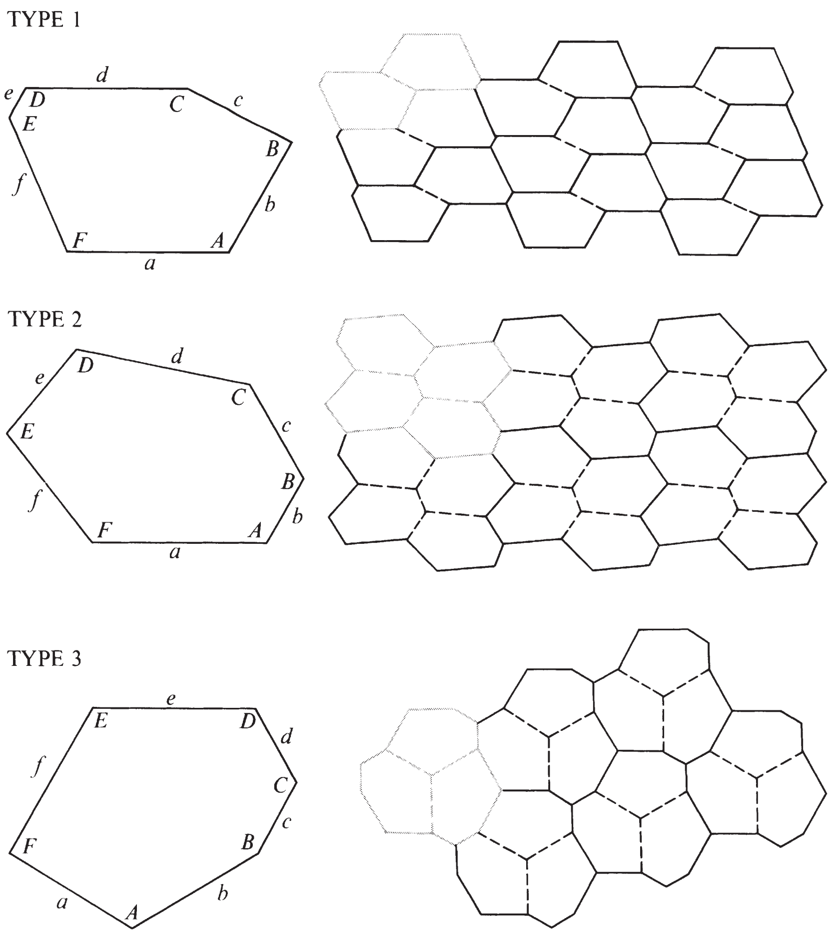


Figure 79 The three types of convex hexagon tiler

"For reasons that I would have difficulty explaining I have been intrigued by this problem for some thirty-five years. Every five or ten years I have made some kind of attempt to solve the problem. Some two years ago I finally discovered a method of classifying the possibilities for pentagons in a more convenient way than Reinhardt's to yield an approach that was humanly possible to carry to completion (though just barely). The result of this investigation was the discovery that there were just three additional types of pentagon . . . that can pave the plane. These pavings are totally surprising. The discovery of their existence is a source of considerable gratification."

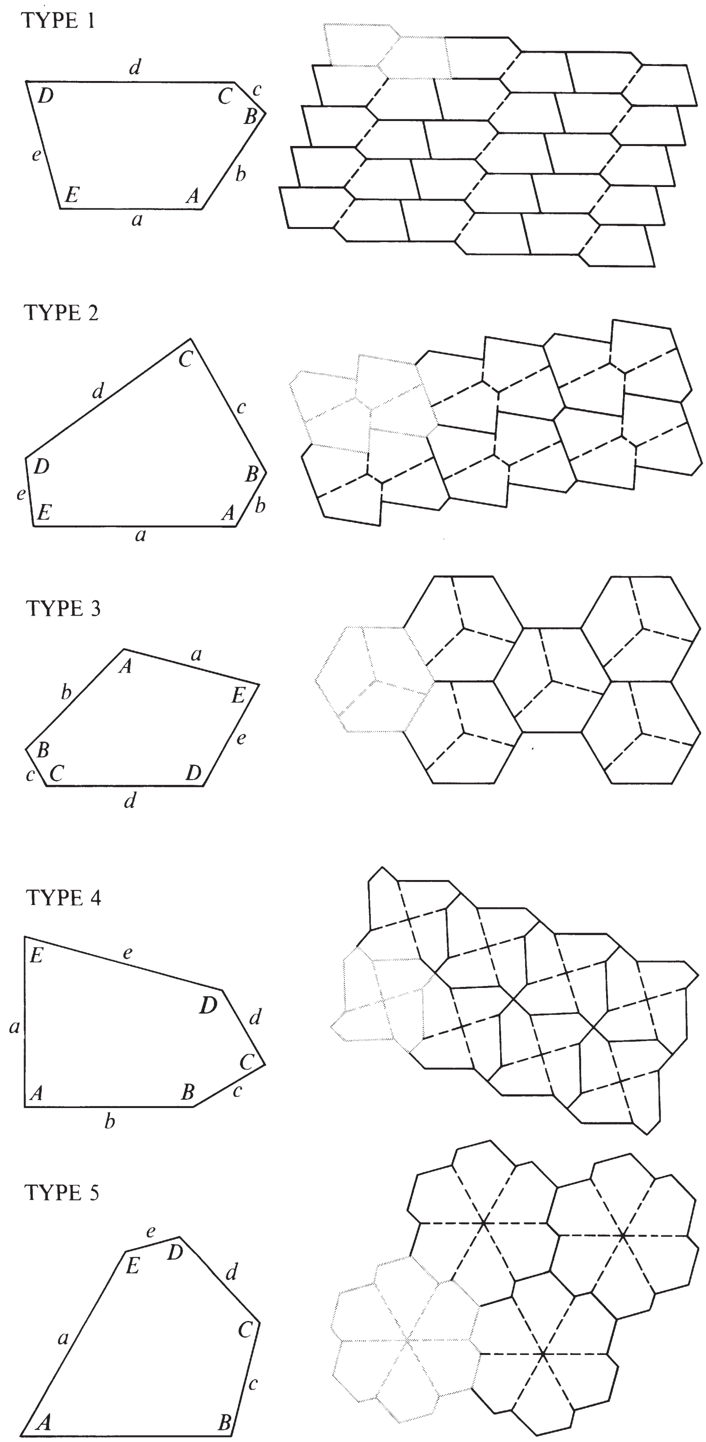


Figure 80 The five types of tiling convex pentagon known in 1918

The three additional types (*see* Figure 81) are described by Kershner as follows (Type 7 and Type 8 require reflection):

6. $A + B + D = 360°$, $A = 2C$,
   and $a = b = e$, $c = d$.
7. $2B + C = 2D + A = 360°$,
   and $a = b = c = d$.
8. $2A + B = 2D + C = 360°$,
   and $a = b = c = d$.

Kershner's paper does not include a proof that there are no other convex pentagons that tile the plane, "for the excellent reason," reads the editor's introductory note, "that a complete proof would require a rather large book."

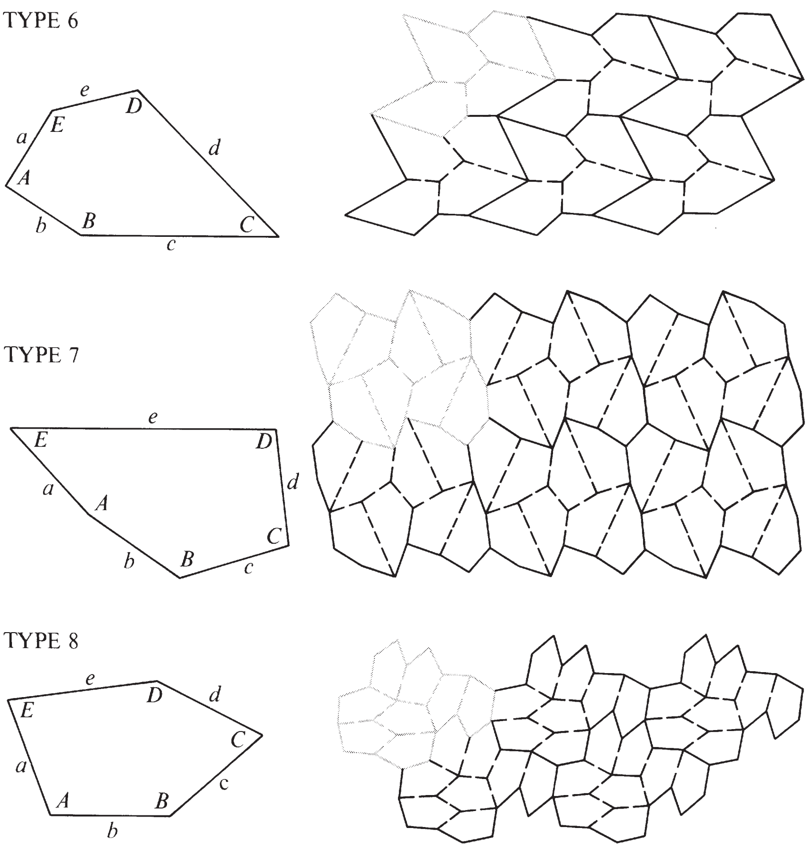


Figure 81 Three new types of tiling convex pentagon discovered in 1967

Note that Kershner has deliberately drawn his tessellations with polygons that are as irregular as possible, within the limits of their type, in order to bring out the nature of the tessellation. The most regular hexagonal tessellation is, of course, the familiar beehive mosaic. One can readily see that it belongs to all three hexagon types.

If the beehive hexagons are bisected, the result is a pentagonal pattern that belongs to Type 1(*see* Figure 82*A*). The pattern formed by six pentagons in a flowerlike arrangement (*see* Figure 82*B*) uses a tile that belongs to Type 1, Type 5, and Type 6. The most remarkable of all the pentagonal patterns is a tessellation of *equilateral* pentagons (*see* Figure 82*C*). It belongs to Types 2 and 4. Observe how quadruplets of these pentagons can be grouped into oblong hexagons in two different ways, each set tessellating the plane at right angles to the other. This beautiful tessellation is frequently seen as a street tiling in Cairo and occasionally in the mosaics of Moorish buildings. It underlies many of Escher's tessellations.

The equilateral pentagon is readily constructed with a compass and a straightedge (*see* Figure 83). First draw a side of the pentagon, *AB.* Construct its perpendicular bisector, *CD,* then draw lines *CE* and *CF* at 45 degrees to *AB.* With center at *A* and radius *AB,* draw a circular arc cutting *CE* at *P.* The same construction is repeated on the other side, with center at *B* and the arc cutting *CF* at *R.* Keeping the compass with radius *AB,* let *R* be the center and strike an arc that cuts the perpendicular bisector, *CD,* at *Q.*

The pentagon's corner angles at *P* and *R* are right angles. The corner at *Q* is a little more than 131 degrees, and corners *A* and *B* are a trifle more than 114 degrees. The length from *Q* to *B* is the product of a side of the pentagon and the square root of two. The pentagon's area (it is easy to prove) is precisely the square of line segment *CR.*

Among the infinite tessellations of the plane that can be made with congruent nonconvex polygons, combinatorial geometers have given special attention in recent years to tiling with polyominoes and their cousins the polyiamonds and polyhexes. (Polyominoes are formed by joining unit squares, polyiamonds by joining equilateral triangles, and polyhexes by joining regular hexagons.) Many fascinating problems have been formulated, some solved and some not. That will be the topic of the next chapter.

ADDENDUM

A remarkable letter on this chapter after it first appeared as a column in 1975, came from Richard E. James III, a computer scientist with the Control Data Corporation. He sent a strange tessellation (*see* Figure 84) along with a note

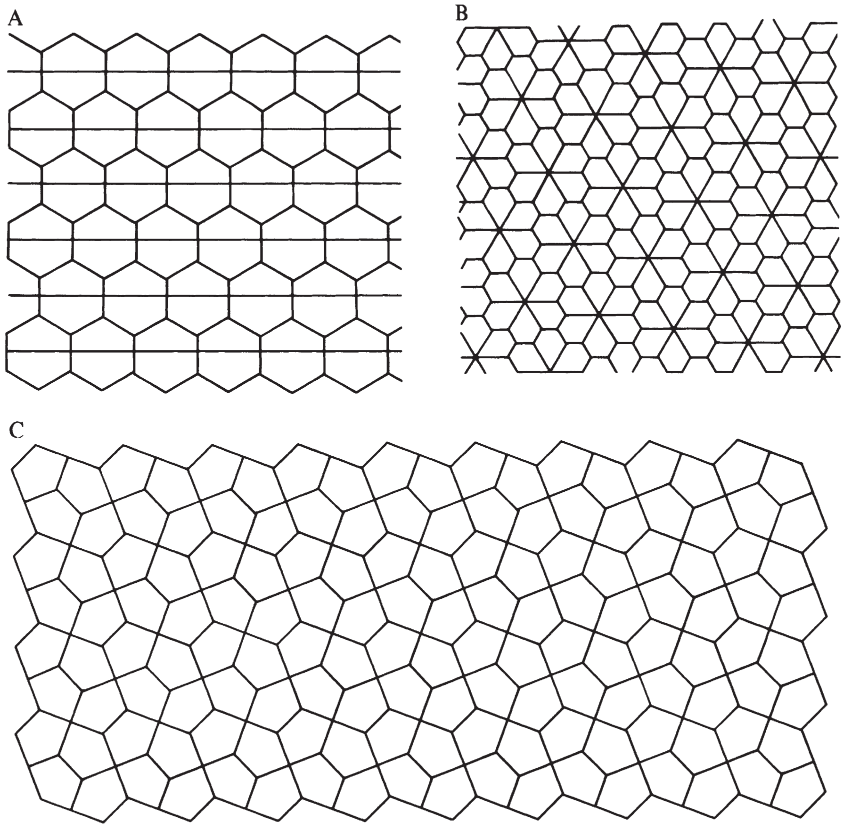


Figure 82 Pentagonal tessellations of unusual symmetry

describing the pentagon (in R. B. Kershner's notation) as $A = 90$ degrees, $C + D = 270$ degrees, $2D + E = 2C + B = 360$ degrees, and $a = b = c + e$. "Do you agree that Kershner missed this one?" he asked.

Kershner had indeed missed it. This means that the problem of classifying all convex pentagons that tile the plane is not solved, as Kershner had supposed. I must say that Kershner received the blow with grace and good humor. In a letter to him, I mentioned that James's discovery illustrates the pragmatic side of mathematical proof, namely that proofs are not known to be proofs until there is a consensus among experts. Kershner replied as follows:

"In connection with your philosophical comments on the nature of a proof you might be interested in an observation by that eminent authority, me. In *The Anatomy of Mathematics,* Kershner and L. R. Wilcox (Ronald Press, 1950) I wrote:

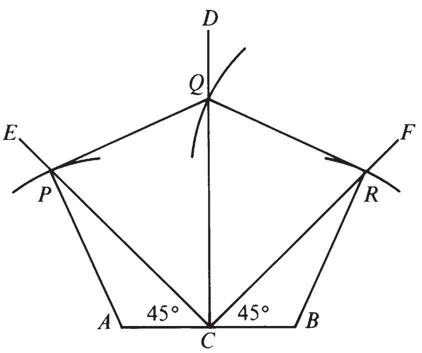


Figure 83 How to construct equilateral-pentagon tiler

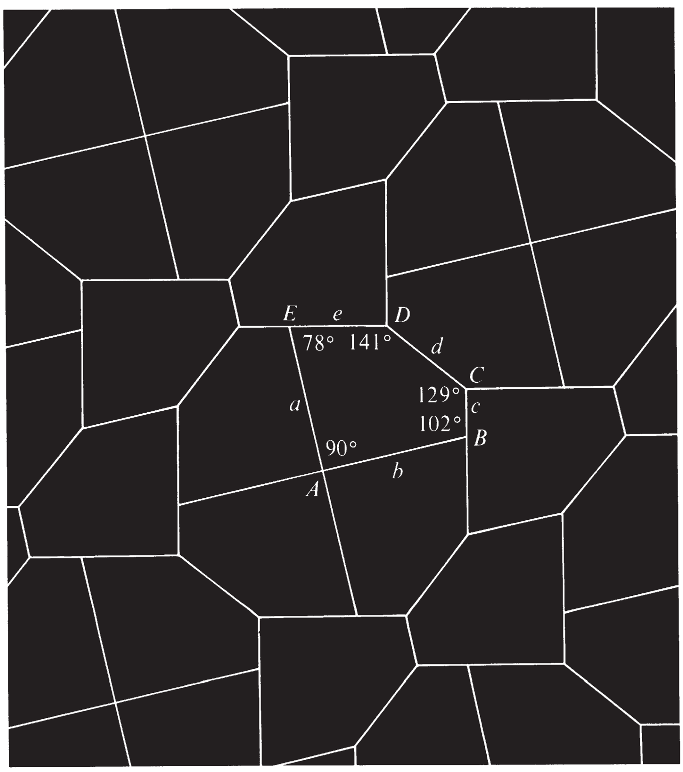


Figure 84 A remarkable new tessellation with congruent convex pentagons by Richard E. James III

> Now it must be said that there is no simple test that can be applied to determine the validity of a proof, that is, to determine that an alleged proof really is a proof. Mathematical history contains rare instances of arguments that were generally accepted as proofs for hundreds of years, before being successfully challenged by a very ingenious mathematician, who pointed out a possibility that had been overlooked in the alleged proof. And more recently, every year there appear, in the mathematical journals of the world, a certain number of papers which point out that some statement, allegedly proved in a preceding paper, was not only erroneously proved (that is, not proved) but was, in fact, incorrect. These facts are mentioned for the benefit of those who feel that there is some magic formula for a proof which makes it immutable and unarguable henceforth and forevermore.

"I must say that when I wrote this paragraph I did not at the time propose eventually to illustrate its validity so graphically myself."

James's tessellation can be varied in ways that have been analyzed in a 1978 paper by Doris Schattschneider of Moravian College, Bethlehem, Pa. It is a basic pattern that could have been discovered by the medieval Moors or even by the ancient Greeks or Romans, but it is probable that the pattern had never before been seen by human eyes until James first put it on paper!

The discovery of new types of tiling pentagons did not end with James's finding. Marjorie Rice, a San Diego housewife with no mathematical training beyond the minimum required in high school, began a systematic search for new patterns. In 1976 she discovered a tenth type (*see* Figure 85), two more types later that year, and still another one the following year, bringing the total number of types to thirteen. A fourteenth type was found in 1985 by Rolf Stein, a mathematics graduate student at the University of Dortmund in West Germany. Its tiling pattern is on the cover of *Mathematics Magazine* (November 1985), with a note about it on page 308. As far as I know, no more new types have been discovered, although there is as yet no proof that the list is complete. Nor is there a full listing of all nonconvex pentagons that tile the plane.

Doris Schattschneider gives a brief account of Mrs. Rice's fantastic achievements in her 1978 paper, and a more detailed account in her contribution to *The Mathematical Gardner.* The latter paper includes three color plates of beautiful Escher-like tesselations (bees, fish, and flowers) that Mrs. Rice based on her new tiling patterns, and a color plate of a handwoven rug based on the James tesselation. The bee pattern provides the book's jacket.

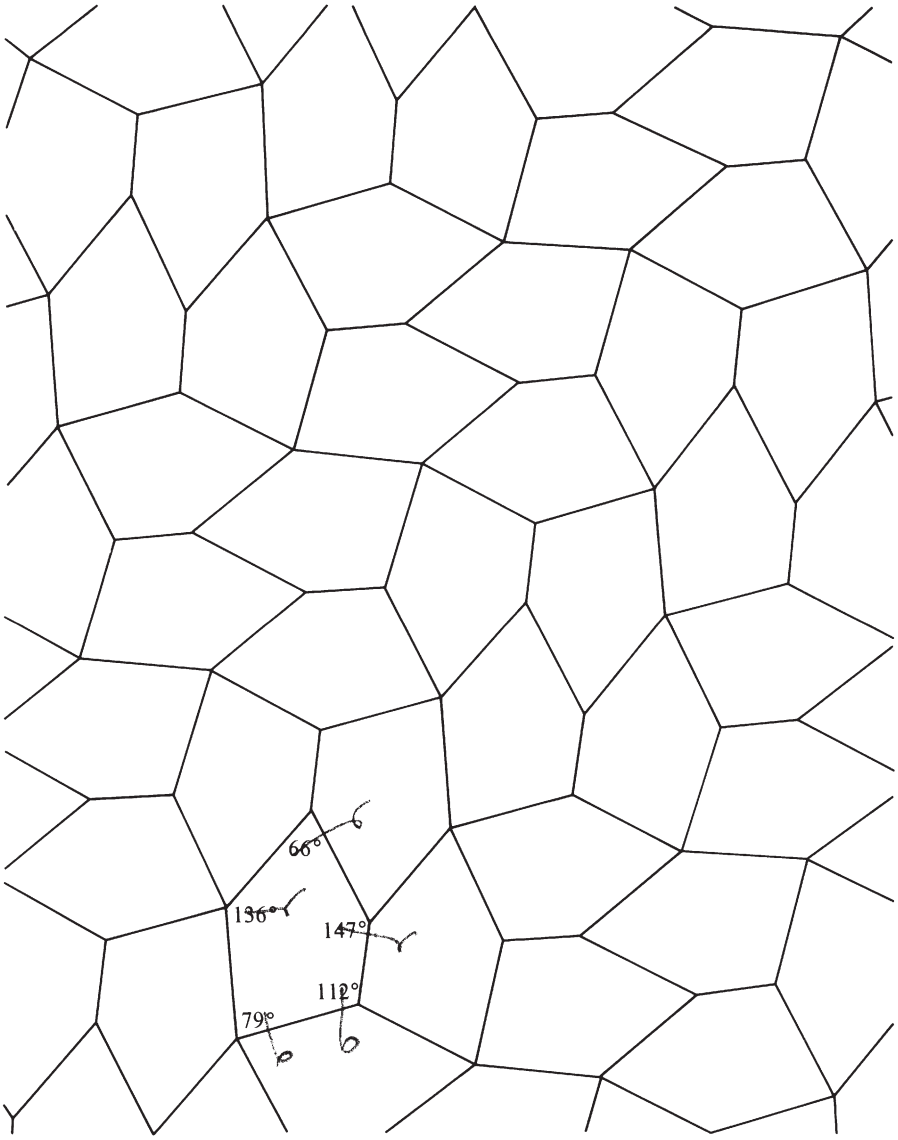


Figure 85 Marjorie Rice's tenth pentagonal tiling

BIBLIOGRAPHY

"On Paving the Plane." R. B. Kershner, in *The American Mathematical Monthly* 75, October 1968, pp. 839–844.

"On Paving the Plane." R. B. Kershner, in *APL Technical Digest* 8, July–August 1969, pp. 4–10.

"The Laws of Sines and Cosines for Polygons." R. B. Kershner, in *Mathematics Magazine* 44, May 1971, pp. 150–153.

"Tesselations with Pentagons." J. A. Dunn, in *The Mathematical Gazette* 55, December 1971, pp. 366–369; see also 56, December 1972, pp. 332–335.

"Tesselations of Pentagons." Stanley R. Clemens, in *Mathematics Teaching* (a British journal for elementary teachers), June 1974, pp. 18–19. See follow-up by John Parker, *ibid.* March 1975, p. 34.

"Tiling the Plane with Congruent Pentagons." Doris Schattschneider, in *Mathematics Magazine* 51, January 1978, pp. 29–44. Reprinted in *Percy Alexander MacMahon: Collected Papers,* Vol. 2, George E. Andrews, ed. MIT Press, 1986.

"Convex Polygons that Cannot Tile the Plane." Ivan Niven, in *The American Mathematical Monthly* 85, December 1978, pp. 785–789.

"A Note on a Result of Niven on Impossible Tesselations." M. S. Klamkin and A. Liu, in *The American Mathematical Monthly* 87, October 1980, pp. 651–653.

"In Praise of Amateurs." Doris Schattschneider, in *The Mathematical Gardner,* David A. Klarner, ed. Prindle, Weber and Schmidt, 1981.

"Equilateral Convex Pentagons Which Tile the Plane." M. D. Hirschhorn and D. C. Hunt, in *Journal of Combinatorial Theory* 39, May 1985, pp. 1–18.

"Tilings by Polygons." Chapter 9 of *Tilings and Patterns.* Branko Grünbaum and Geoffrey C. Shephard. W. H. Freeman and Company, 1987. This marvelous 700-page work is the first comprehensive treatise on all aspects of tiling theory.

Tiling with Nonconvex Pentagons

"Tesselations with Equilateral Reflex Polygons." Gillian Hatch and David Simonds, in *Mathematics Teaching* September 1978, pp. 32–34.

"Tesselations of Non-convex Pentagons." Michael Palmer, *ibid,* March 1979, pp. 8–11.

"Spiral Tilings and Versatiles." Branko Grünbaum and G. C. Shephard, *ibid,* September 1979, pp. 50–51.

"The Incredible Pentagonal Versatile." Doris Schattschneider, designs by Marjorie Rice, *ibid,* December 1980, pp. 52–53.

Solution to Problem 67D, posed by H. Martyn Cundy, based on results by Doris Schattschneider, in *The Mathematical Gazette* 67, December 1983, pp. 307–309.