CHAPTER 1
INTRODUCTION TO MODERN NETWORK THEORY

1.1 MODERN NETWORK THEORY

A generalized filter is shown in Figure 1-1. The filter block may consist of inductors, capacitors, resistors, and possibly active elements such as operational amplifiers and transistors. The terminations shown are a voltage source E_s, a source resistance R_s, and a load resistor R_L.

The circuit equations for the network of Figure 1-1 can be written by using circuit-analysis techniques. Modern network theory solves these equations to determine the network values for optimum performance in some respect.

The Pole-Zero Concept

The frequency response of the generalized filter can be expressed as a ratio of two polynomials in s where $s = j\omega$ ($j = \sqrt{-1}$, and ω, the frequency in radians per second, is $2\pi f$) and is referred to as a transfer function. This can be stated mathematically as

$$T(s) = \frac{E_L}{E_s} = \frac{N(s)}{D(s)} \tag{1-1}$$

The roots of the denominator polynomial $D(s)$ are called poles and the roots of the numerator polynomial $N(s)$ are referred to as zeros.

Deriving a network's transfer function could become quite tedious and is beyond the scope of this book. The following discussion explores the evaluation and representation of a relatively simple transfer function.

Analysis of the low-pass filter of Figure 1-2*a* results in the following transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

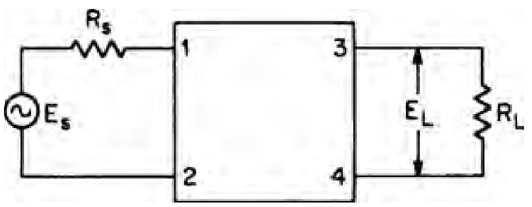

FIGURE 1-1 A generalized filter.

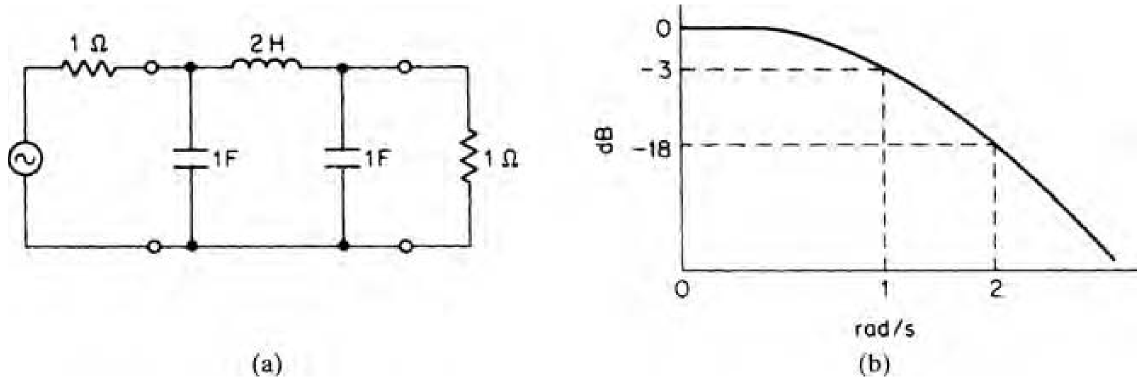

FIGURE 1-2 An all-pole $n = 3$ low-pass filter: (*a*) a filter circuit; and (*b*) a frequency response.

Let us now evaluate this expression at different frequencies after substituting $j\omega$ for s. The result will be expressed as the absolute magnitude of $T(j\omega)$ and the relative attention in decibels with respect to the response at DC.

$$T(j\omega) = \frac{1}{1 - 2\omega^2 + j(2\omega - \omega^3)} \tag{1-3}$$

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0	1	0 dB
1	0.707	−3 dB
2	0.124	−18 dB
3	0.0370	−29 dB
4	0.0156	−36 dB

The frequency-response curve is plotted in Figure 1-2*b*.

Analysis of Equation (1-2) indicates that the denominator of the transfer function has three roots or poles and the numerator has none. The filter is therefore called an all-pole type. Since the denominator is a third-order polynomial, the filter is also said to have an $n = 3$ complexity. The denominator poles are $s = -1$, $s = -0.500 + j0.866$, and $s = -0.500 - j0.866$.

These complex numbers can be represented as symbols on a complex-number plane. The abscissa is α, the real component of the root, and the ordinate is β, the imaginary part. Each pole is represented as the symbol X, and a zero is represented as 0. Figure 1-3 illustrates the complex-number plane representation for the roots of Equation (1-2).

Certain mathematical restrictions must be applied regarding the location of poles and zeros in order for the filter to be realizable. They must occur in pairs which are conjugates of each other, except for real-axis poles and zeros, which may occur singly. Poles must also be restricted to the left plane (in other words, the real coordinate of the pole must be negative), while zeros may occur in either plane.

Synthesis of Filters from Polynomials. Modern network theory has produced families of standard transfer functions that provide optimum filter performance in some desired respect. Synthesis is the process of deriving circuit component values from these transfer functions. Chapter 11 contains extensive tables of transfer functions and their associated component values so that design by synthesis is not required. Also, computer programs on the CD-ROM simplify the design process. However, in order to gain some understanding

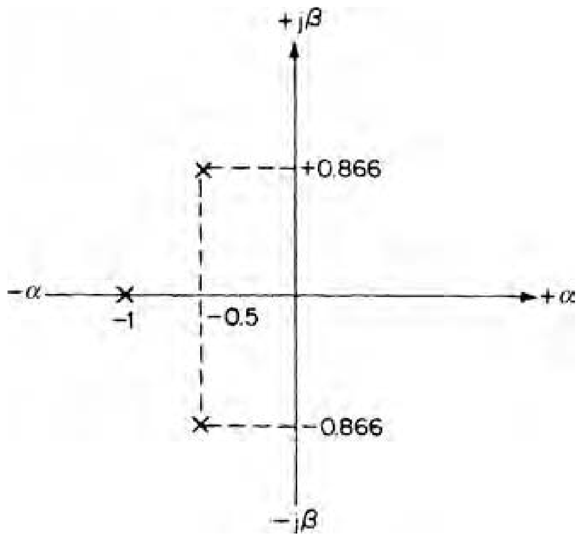

FIGURE 1-3 A complex-frequency plane representation of Equation (1-2).

as to how these values have been determined, we will now discuss a few methods of filter synthesis.

Synthesis by Expansion of Driving-Point Impedance. The input impedance to the generalized filter of Figure 1-1 is the impedance seen looking into terminals 1 and 2 with terminals 3 and 4 terminated, and is referred to as the driving-point impedance or Z_{11} of the network. If an expression for Z_{11} could be determined from the given transfer function, this expression could then be expanded to define the filter.

A family of transfer functions describing the flattest possible shape and a monotonically increasing attenuation in the stopband is known as the *Butterworth low-pass response*. These all-pole transfer functions have denominator polynomial roots, which fall on a circle having a radius of unity from the origin of the $j\omega$ axis. The attenuation for this family is 3 dB at 1 rad/s.

The transfer function of Equation (1-2) satisfies this criterion. It is evident from Figure 1-3 that if a circle were drawn having a radius of 1, with the origin as the center, it would intersect the real root and both complex roots.

If R_s in the generalized filter of Figure 1-1 is set to 1 Ω, a driving-point impedance expression can be derived in terms of the Butterworth transfer function as

$$Z_{11} = \frac{D(s) - s^n}{D(s) + s^n} \tag{1-4}$$

where $D(s)$ is the denominator polynomial of the transfer function and n is the order of the polynomial.

After $D(s)$ is substituted into Equation (1-4), Z_{11} is expanded using the continued fraction expansion. This expansion involves successive division and inversion of a ratio of two polynomials. The final form contains a sequence of terms, each alternately representing a capacitor and an inductor and finally the resistive termination. This procedure is demonstrated by the following example.

Example 1-1 Synthesis of $N = 3$ Butterworth Low-Pass Filter by Continued Fraction Expansion

Required:

A low-pass LC filter having a Butterworth $n = 3$ response.

Result:

(a) Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

(b) Substitute $D(s) = s^3 + 2s^2 + 2s + 1$ and $s^n = s^3$ into Equation (1-4), which results in

$$Z_{11} = \frac{2s^2 + 2s + 1}{2s^3 + 2s^2 + 2s + 1} \tag{1-4}$$

(c) Express Z_{11} so that the denominator is a ratio of the higher-order to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{2s^3 + 2s^2 + 2s + 1}{2s^2 + 2s + 1}}$$

(d) Dividing the denominator and inverting the remainder results in

$$Z_{11} = \cfrac{1}{s + \cfrac{1}{\dfrac{2s^2 + 2s + 1}{s + 1}}}$$

(e) After further division and inversion, we get as our final expression:

$$Z_{11} = \cfrac{1}{s + \cfrac{1}{2s + \cfrac{1}{s + 1}}} \tag{1-5}$$

The circuit configuration of Figure 1-4 is called a ladder network, since it consists of alternating series and shunt branches. The input impedance can be expressed as the following continued fraction:

$$Z_{11} = \cfrac{1}{Y_1 + \cfrac{1}{Z_2 + \cfrac{1}{Y_3 + \cdots \cfrac{1}{Z_{n-1} + \cfrac{1}{Y_n}}}}} \tag{1-6}$$

where $Y = sC$ and $Z = sL$ for the low-pass all-pole ladder except for a resistive termination where $Y_n = sC + 1/R_L$.

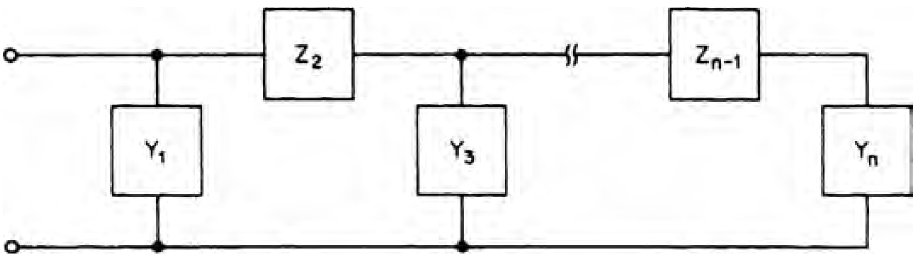

FIGURE 1-4 A general ladder network.

Figure 1-5 can then be derived from Equation (1-5) and (1-6) by inspection. This can be proved by reversing the process of expanding Z_{11}. By alternately adding admittances and impedances while working toward the input, Z_{11} is verified as being equal to Equation (1-5).

Synthesis for Unequal Terminations. If the source resistor is set equal to 1 Ω and the load resistor is desired to be infinite (unterminated), the impedance looking into terminals 1 and 2 of the generalized filter of Figure 1-1 can be expressed as

$$Z_{11}\frac{D(s\text{ even})}{D(s\text{ odd})} \tag{1-7}$$

$D(s$ even) contains all the even-power s terms of the denominator polynomial and $D(s$ odd) consist of all the odd-power s terms of any realizable all-pole low-pass transfer function. Z_{11} is expanded into a continued fraction, as in Example 1-1, to define the circuit.

Example 1-2 Synthesis of $N = 3$ Butterworth Low-Pass Filter for an Infinite Termination

Required:
Low-pass filter having a Butterworth $n = 3$ response with a source resistance of 1 Ω and an infinite termination.

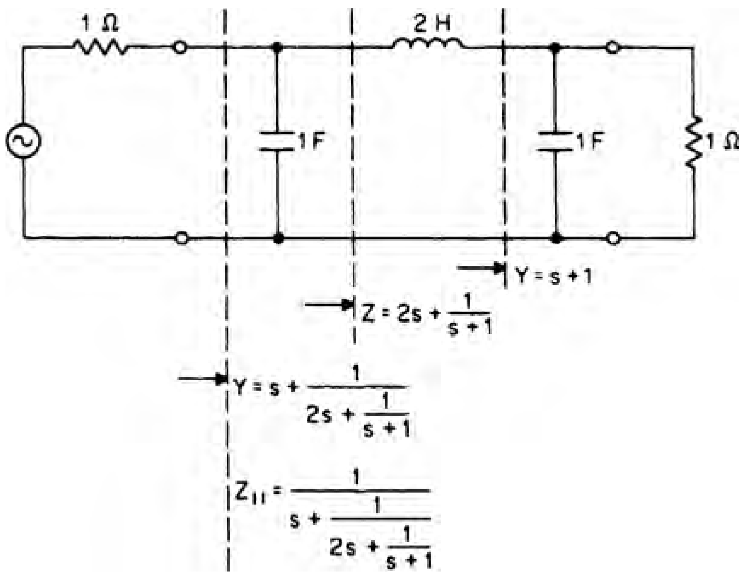

FIGURE 1-5 The low-pass filter for Equation (1-5).

Result:

(a) Use the Butterworth transfer function:

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

(b) Substitute $D(s \text{ even}) = 2s^2 + 1$ and $D(s \text{ odd}) = s^3 + 2s$ into Equation (1-7):

$$Z_{11} = \frac{2s^2 + 1}{s^3 + 2s} \tag{1-7}$$

(c) Express Z_{11} so that the denominator is a ratio of the higher- to the lower-order polynomial:

$$Z_{11} = \frac{1}{\dfrac{s^3 + 2s}{2s^2 + 1}}$$

(d) Dividing the denominator and inverting the remainder results in

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{\dfrac{2s^2 + 1}{1.5s}}}$$

(e) Dividing and further inverting results in the final continued fraction:

$$Z_{11} = \frac{1}{0.5s + \dfrac{1}{1.333s + \dfrac{1}{1.5s}}} \tag{1-8}$$

The circuit is shown in Figure 1-6.

Synthesis by Equating Coefficients. An active three-pole low-pass filter is shown in Figure 1-7. Its transfer function is given by

$$T(s) = \frac{1}{s^3A + s^2B + sC + 1} \tag{1-9}$$

where

$$A = C_1C_2C_3 \tag{1-10}$$

$$B = 2C_3(C_1 + C_2) \tag{1-11}$$

and

$$C = C_2 + 3C_3 \tag{1-12}$$

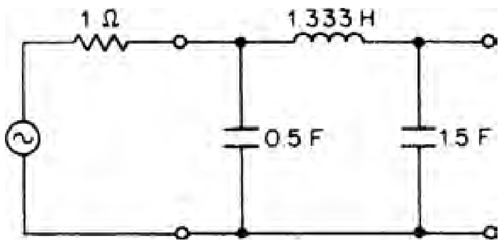

FIGURE 1-6 The low-pass filter of Example 1-2.

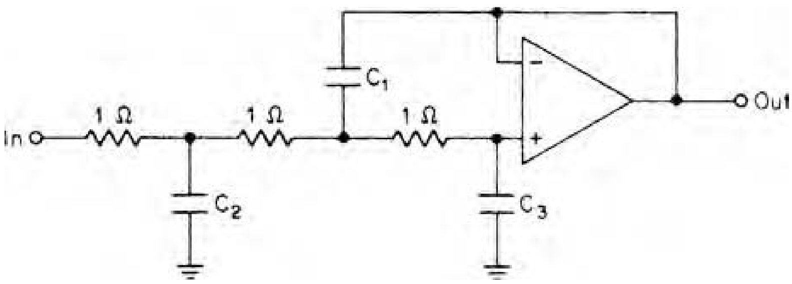

FIGURE 1-7 The general $n = 3$ active low-pass filter.

If a Butterworth transfer function is desired, we can set Equation (1-9) equal to Equation (1-2).

$$T(s) = \frac{1}{s^3A + s^2B + sC + 1} = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-13}$$

By equating coefficients, we obtain

$$A = 1$$
$$B = 2$$
$$C = 2$$

Substituting these coefficients in Equation (1-10) through (1-12) and solving for C_1, C_2, and C_3 results in the circuit of Figure 1-8.

Synthesis of filters directly from polynomials offers an elegant solution to filter design. However, it also may involve laborious computations to determine circuit element values. Design methods have been greatly simplified by the curves, tables, computer programs, and step-by-step procedures provided in this handbook, so design by synthesis can be left to the advanced specialist.

Active vs. Passive Filters. The LC filters of Figures 1-5 and 1-6 and the active filter of Figure 1-8 all satisfy an $n = 3$ Butterworth low-pass transfer function. The filter designer is frequently faced with the sometimes difficult decision of choosing whether to use an active or LC design. A number of factors must be considered. Some of the limitations and considerations for each filter type will now be discussed.

Frequency Limitations. At subaudio frequencies, LC filter designs require high values of inductance and capacitance along with their associated bulk. Active filters are more practical because they can be designed at higher impedance levels so that capacitor magnitudes are reduced.

Above 20 MHz or so, most commercial-grade operational amplifiers have insufficient open-loop gain for the average active filter requirement. However, amplifiers are available

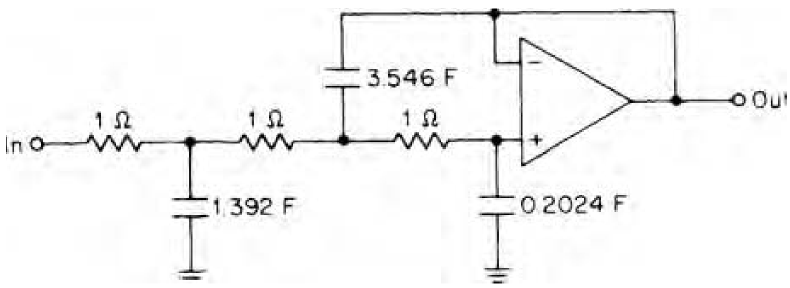

FIGURE 1-8 A Butterworth $n = 3$ active low-pass filter.

with extended bandwidth at an increased cost so that active filters at frequencies up to 100 MHz are possible. LC filters, on the other hand, are practical at frequencies up to a few hundred megahertz. Beyond this range, filters become impractical to build in lumped form, and so distributed parameter techniques are used, such as stripline or microstrip, where a PC board functions as a distributed transmission line.

Size Considerations. Active filters are generally smaller than their LC counterparts since inductors are not required. Further reduction in size is possible with microelectronic technology. Surface mount components for the most part have replaced Hybrid technology, whereas in the past Hybrids were the only way to reduce the size of active filters.

Economics and Ease of Manufacture. LC filters generally cost more than active filters because they use inductors. High-quality coils require efficient magnetic cores. Sometimes, special coil-winding methods are needed as well. These factors lead to the increased cost of LC filters.

Active filters have the distinct advantage that they can be easily assembled using standard off-the-shelf components. LC filters require coil-winding and coil-assembly skills. In addition, eliminating inductors prevents magnetic emissions, which can be troublesome.

Ease of Adjustment. In critical LC filters, tuned circuits require adjustment to specific resonances. Capacitors cannot be made variable unless they are below a few hundred picofarads. Inductors, however, can easily be adjusted, since most coil structures provide a means for tuning, such as an adjustment slug for a Ferrite potcore.

Many active filter circuits are not easily adjustable, however. They may contain RC sections where two or more resistors in each section have to be varied in order to control resonance. These types of circuit configurations are avoided. The active filter design techniques presented in this handbook include convenient methods for adjusting resonances where required, such as for narrowband bandpass filters.

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CHAPTER 2
SELECTING THE RESPONSE CHARACTERISTIC

2.1 FREQUENCY-RESPONSE NORMALIZATION

Several parameters are used to characterize a filter's performance. The most commonly specified requirement is frequency response. When given a frequency-response specification, the engineer must select a filter design that meets these requirements. This is accomplished by transforming the required response to a normalized low-pass specification having a cutoff of 1 rad/s. This normalized response is compared with curves of normalized low-pass filters which also have a 1-rad/s cutoff. After a satisfactory low-pass filter is determined from the curves, the tabulated normalized element values of the chosen filter are transformed or denormalized to the final design.

Modern network theory has provided us with many different shapes of amplitude versus frequency which have been analytically derived by placing various restrictions on transfer functions. The major categories of these low-pass responses are

- Butterworth
- Chebyshev
- Linear Phase
- Transitional
- Synchronously tuned
- Elliptic-function

With the exception of the elliptic-function family, these responses are all normalized to a 3-dB cutoff of 1 rad/s.

Frequency and Impedance Scaling

The basis for normalization of filters is the fact that a given filter's response can be scaled (shifted) to a different frequency range by dividing the reactive elements by a frequency-scaling factor (FSF). The FSF is the ratio of a reference frequency of the desired response to the corresponding reference frequency of the given filter. Usually 3-dB points are selected as reference frequencies of low-pass and high-pass filters, and the center frequency is chosen as the reference for bandpass filters. The FSF can be expressed as

$$\text{FSF} = \frac{\text{desired reference frequency}}{\text{existing reference frequency}} \tag{2-1}$$

The FSF must be a dimensionless number; so both the numerator and denominator of Equation (2-1) must be expressed in the same units, usually radians per second. The following example demonstrates the computation of the FSF and frequency scaling of filters.

Example 2-1 Frequency Scaling of a Low-Pass Filter

Required:
A low-pass filter, either *LC* or active, with an $n = 3$ Butterworth transfer function having a 3-dB cutoff at 1000 Hz.

Result:
Figure 2-1 illustrates the *LC* and active $n = 3$ Butterworth low-pass filters discussed in Chapter 1 and their response.

(*a*) Compute FSF.

$$\text{FSF} = \frac{2\pi 1000 \text{ rad/s}}{1 \text{ rad/s}} = 6280 \tag{2-1}$$

(*b*) Dividing all the reactive elements by the FSF results in the filters of Figure 2-2*a* and *b* and the response of Figure 2-2*c*.

Note that all points on the frequency axis of the normalized response have been multiplied by the FSF. Also, since the normalized filter has its cutoff at 1 rad/s, the FSF can be directly expressed by $2\pi f_c$, where f_c is the desired low-pass cutoff frequency in hertz.

Frequency scaling a filter has the effect of multiplying all points on the frequency axis of the response curve by the FSF. Therefore, a normalized response curve can be directly used to predict the attenuation of the denormalized filter.

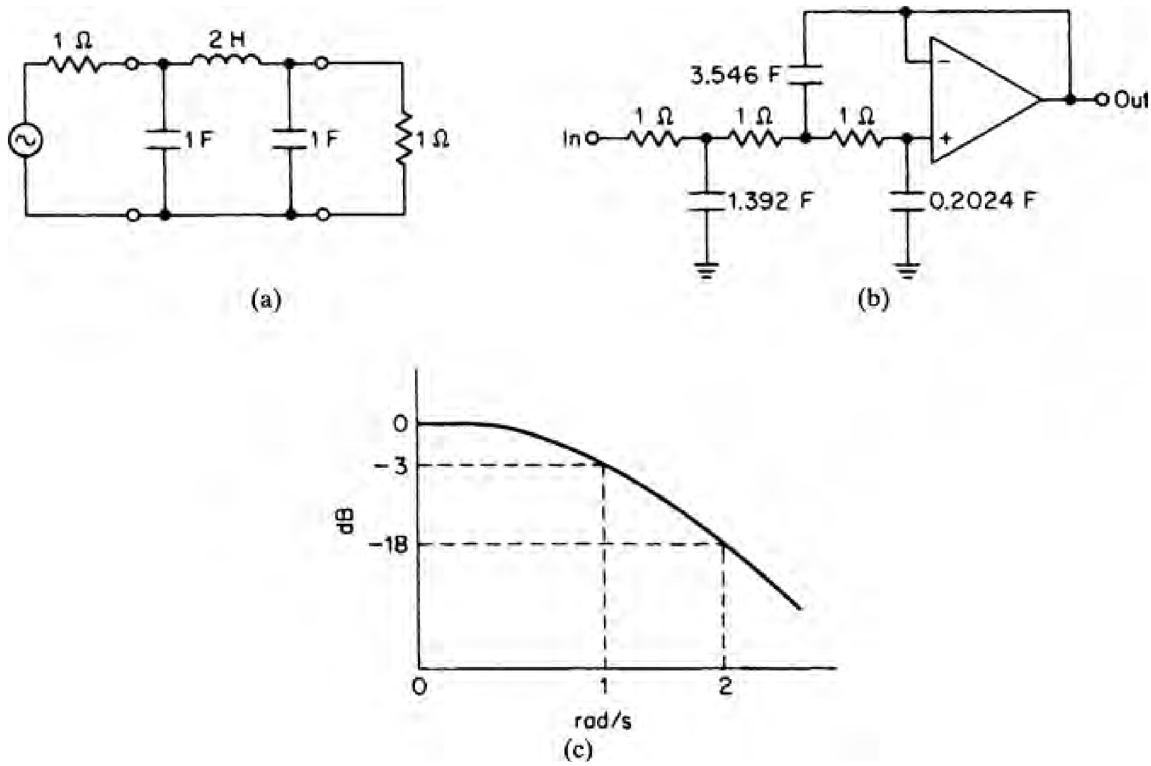

FIGURE 2-1 $n = 3$ Butterworth low-pass filter: (*a*) *LC* filter; (*b*) active filter; and (*c*) frequency response.

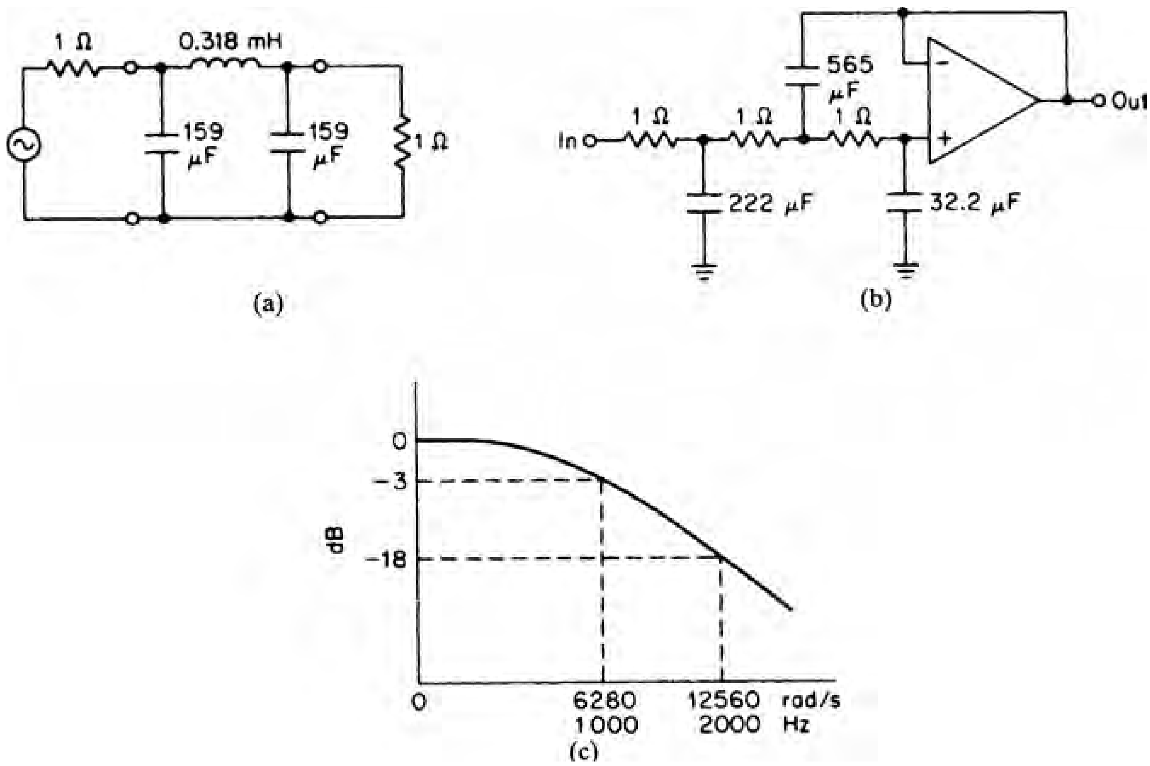

FIGURE 2-2 The denormalized low-pass filter of Example 2-1: (*a*) *LC* filter; (*b*) active filter; and (*c*) frequency response.

When the filters of Figure 2-1 were denormalized to those of Figure 2-2, the transfer function changed as well. The denormalized transfer function became

$$T(s) = \frac{1}{4.03 \times 10^{-12}s^3 + 5.08 \times 10^{-9}s^2 + 3.18 \times 10^{-4}s + 1} \tag{2-2}$$

The denominator has roots:

$$s = -6280,\ s = -3140 + j5438, \text{ and } s = -3140 - j5438.$$

These roots can be obtained directly from the normalized roots by multiplying the normalized root coordinates by the FSF. Frequency scaling a filter also scales the poles and zeros (if any) by the same factor.

The component values of the filters in Figure 2-2 are not very practical. The capacitor values are much too large and the 1-Ω resistor values are not very desirable. This situation can be resolved by impedance scaling. Any linear active or passive network maintains its transfer function if all resistor and inductor values are multiplied by an impedance-scaling factor Z, and all capacitors are divided by the same factor Z. This occurs because the Zs cancel in the transfer function. To prove this, let's investigate the transfer function of the simple two-pole low-pass filter of Figure 2-3*a*, which is

$$T(s) = \frac{1}{s^2LC + sCR + 1} \tag{2-3}$$

Impedance scaling can be mathematically expressed as

$$R' = ZR \tag{2-4}$$

$$L' = ZL \tag{2-5}$$

$$C' = \frac{C}{Z} \tag{2-6}$$

where the primes denote the values after impedance scaling.

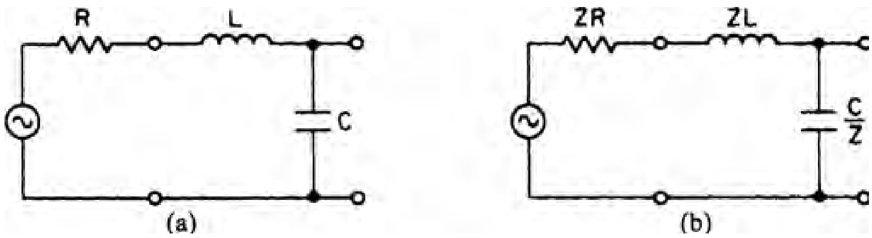

FIGURE 2-3 A two-pole low-pass *LC* filter: (*a*) a basic filter; and (*b*) an impedance-scaled filter.

If we impedance-scale the filter, we obtain the circuit of Figure 2-3*b*. The new transfer function then becomes

$$T(s) = \frac{1}{s^2 ZL\frac{C}{Z} + s\frac{C}{Z}ZR + 1} \tag{2-7}$$

Clearly, the *Z*s cancel, so both transfer functions are equivalent.

We can now use impedance scaling to make the values in the filters of Figure 2-2 more practical. If we use impedance scaling with a *Z* of 1000, we obtain the filters of Figure 2-4. The values are certainly more suitable.

Frequency and impedance scaling are normally combined into one step rather than performed sequentially. The denormalized values are then given by

$$R' = R \times Z \tag{2-8}$$

$$L' = \frac{L \times Z}{\text{FSF}} \tag{2-9}$$

$$C' = \frac{C}{\text{FSF} \times Z} \tag{2-10}$$

where the primed values are both frequency- and impedance-scaled.

Low-Pass Normalization. In order to use normalized low-pass filter curves and tables, a given low-pass filter requirement must first be converted into a normalized requirement. The curves can now be entered to find a satisfactory normalized filter which is then scaled to the desired cutoff.

The first step in selecting a normalized design is to convert the requirement into a steepness factor A_s, which can be defined as

$$A_s = \frac{f_s}{f_c} \tag{2-11}$$

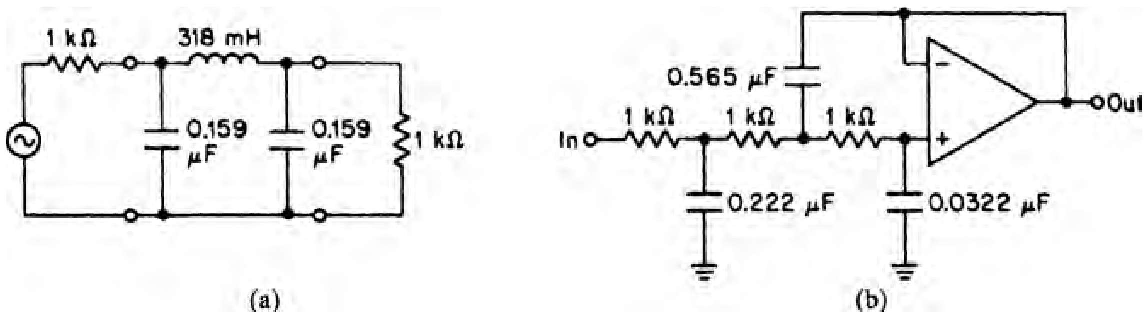

FIGURE 2-4 The impedance-scaled filters of Example 2-1: (*a*) *LC* filter; and (*b*) active filter.

where f_s is the frequency having the minimum required stopband attenuation and f_c is the limiting frequency or cutoff of the passband, usually the 3-dB point. The normalized curves are compared with A_s, and a design is selected that meets or exceeds the requirement. The design is often frequency scaled so that the selected passband limit of the normalized design occurs at f_c.

If the required passband limit f_c is defined as the 3-dB cutoff, the steepness factor A_s can be directly looked up in radians per second on the frequency axis of the normalized curves.

Suppose that we required a low-pass filter that has a 3-dB point at 100 Hz and more than 30-dB attenuation at 400 Hz. A normalized low-pass filter that has its 3-dB point at 1 rad/s and over 30-dB attenuation at 4 rad/s would meet the requirement if the filter were frequency-scaled so that the 3-dB point occurred at 100 Hz. Then there would be over 30-dB attenuation at 400 Hz, or four times the cutoff, because a response shape is retained when a filter is frequency scaled.

The following example demonstrates normalizing a simple low-pass requirement.

Example 2-2 Normalizing a Low-Pass Specification for a 3-dB cutoff

Required:
Normalize the following specification:

A low-pass filter
3 dB at 200 Hz
30-dB minimum at 800 Hz

Result:

(*a*) Compute A_s.

$$A_s = \frac{f_s}{f_c} = \frac{800 \text{ Hz}}{200 \text{ Hz}} = 4 \tag{2-11}$$

(*b*) Normalized requirement:

3 dB at 1 rad/s
30-dB minimum at 4 rad/s

In the event f_c does not correspond to the 3-dB cutoff, A_s can still be computed and a normalized design found that will meet the specifications. This is illustrated in the following example.

Example 2-3 Normalizing a Low-Pass Specification for a 1-dB cutoff

Required:
Normalize the following specification:

A low-pass filter
1 dB at 200 Hz
30-dB minimum at 800 Hz

Result:

(*a*) Compute A_s.

$$A_s = \frac{f_s}{f_c} = \frac{800 \text{ Hz}}{200 \text{ Hz}} = 4 \tag{2-11}$$

(*b*) Normalized requirement:

1 dB at K rad/s

30-dB minimum at 4 K rad/s

(where K is arbitrary)

A possible solution to Example 2-3 would be a normalized filter which has a 1-dB point at 0.8 rad/s and over 30 dB attenuation at 3.2 rad/s. The fundamental requirement is that the normalized filter makes the transition between the passband and stopband limits within a frequency ratio A_s.

High-Pass Normalization. A normalized $n = 3$ low-pass Butterworth transfer function was given in section 1.1 as

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \tag{1-2}$$

and the results of evaluating this transfer function at various frequencies were

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0	1	0 dB
1	0.707	−3 dB
2	0.124	−18 dB
3	0.0370	−29 dB
4	0.0156	−36 dB

Let's now perform a high-pass transformation by substituting $1/s$ for s in Equation (1-2). After some algebraic manipulations, the resulting transfer function becomes

$$T(s) = \frac{s^3}{s^3 + 2s^2 + 2s + 1} \tag{2-12}$$

If we evaluate this expression at specific frequencies, we can generate the following table:

ω	$\lvert T(j\omega)\rvert$	$20 \log \lvert T(j\omega)\rvert$
0.25	0.0156	−36 dB
0.333	0.0370	−29 dB
0.500	0.124	−18 dB
1	0.707	−3 dB
∞	1	0 dB

The response is clearly that of a high-pass filter. It is also apparent that the low-pass attenuation values now occur at high-pass frequencies that are exactly the reciprocals of the corresponding low-pass frequencies. A high-pass transformation of a normalized low-pass filter transposes the low-pass attenuation values to reciprocal frequencies and retains the 3-dB cutoff at 1 rad/s. This relationship is evident in Figure 2-5, where both filter responses are compared.

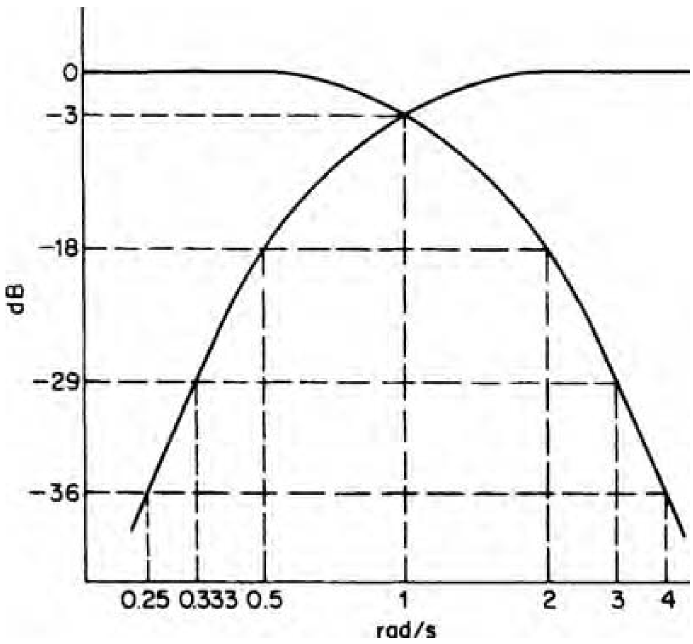

FIGURE 2-5 A normalized low-pass high-pass relationship.

The normalized low-pass curves could be interpreted as normalized high-pass curves by reading the attenuation as indicated and taking the reciprocals of the frequencies. However, it is much easier to convert a high-pass specification into a normalized low-pass requirement and use the curves directly.

To normalize a high-pass filter specification, calculate A_s, which in the case of high-pass filters is given by

$$A_s = \frac{f_c}{f_s} \tag{2-13}$$

Since the A_s, for high-pass filters is defined as the reciprocal of the A_s for low-pass filters, Equation (2-13) can be directly interpreted as a low-pass requirement. A normalized low-pass filter can then be selected from the curves. A high-pass transformation is performed on the corresponding low-pass filter, and the resulting high-pass filter is scaled to the desired cutoff frequency.

The following example shows the normalization of a high-pass filter requirement.

Example 2-4 Normalizing a High-Pass Specification

Required:

Normalize the following requirement:

A high-pass filter
3 dB at 200 Hz
30-dB minimum at 50 Hz

Result:

(*a*) Compute A_s.

$$A_s = \frac{f_c}{f_s} = \frac{200\text{ Hz}}{50\text{ Hz}} = 4 \tag{2-13}$$

(*b*) Normalized equivalent low-pass requirement:

3 dB at 1 rad/s

30-dB minimum at 4 rad/s

Bandpass Normalization. Bandpass filters fall into two categories: narrowband and wideband. If the ratio of the upper cutoff frequency to the lower cutoff frequency is over 2 (an octave), the filter is considered a wideband type.

Wideband Bandpass Filters. Wideband filter specifications can be separated into individual low-pass and high-pass requirements which are treated independently. The resulting low-pass and high-pass filters are then cascaded to meet the composite response.

Example 2-5 Normalizing a Wideband Bandpass Filter

Required:
Normalize the following specification:

bandpass filter

3 dB at 500 and 1000 Hz

40-dB minimum at 200 and 2000 Hz

Result:

(*a*) Determine the ratio of upper cutoff to lower cutoff.

$$\frac{1000\text{ Hz}}{500\text{ Hz}} = 2$$

wideband type

(*b*) Separate requirement into individual specifications.

High-pass filter:	**Low-pass filter:**
3 dB at 500 Hz	3 dB at 1000 Hz
40-dB minimum at 200 Hz	40-dB minimum at 2000 Hz
$A_s = 2.5$ (2-13)	$A_s = 2.0$ (2-11)

(*c*) Normalized high-pass and low-pass filters are now selected, scaled to the required cutoff frequencies, and cascaded to meet the composite requirements. Figure 2-6 shows the resulting circuit and response.

Narrowband Bandpass Filters. Narrowband bandpass filters have a ratio of upper cutoff frequency to lower cutoff frequency of approximately 2 or less and cannot be designed as separate low-pass and high-pass filters. The major reason for this is evident from Figure 2-7. As the ratio of upper cutoff to lower cutoff decreases, the loss at the center frequency will increase, and it may become prohibitive for ratios near unity.

If we substitute $s + 1/s$ for s in a low-pass transfer function, a bandpass filter results. The center frequency occurs at 1 rad/s, and the frequency response of the low-pass filter is directly transformed into the bandwidth of the bandpass filter at points of equivalent attenuation. In other words, the attenuation bandwidth ratios remain unchanged. This is shown in Figure 2-8, which shows the relationship between a low-pass filter and its transformed bandpass equivalent. Each pole and zero of the low-pass filter is transformed into a *pair* of poles and zeros in the bandpass filter.

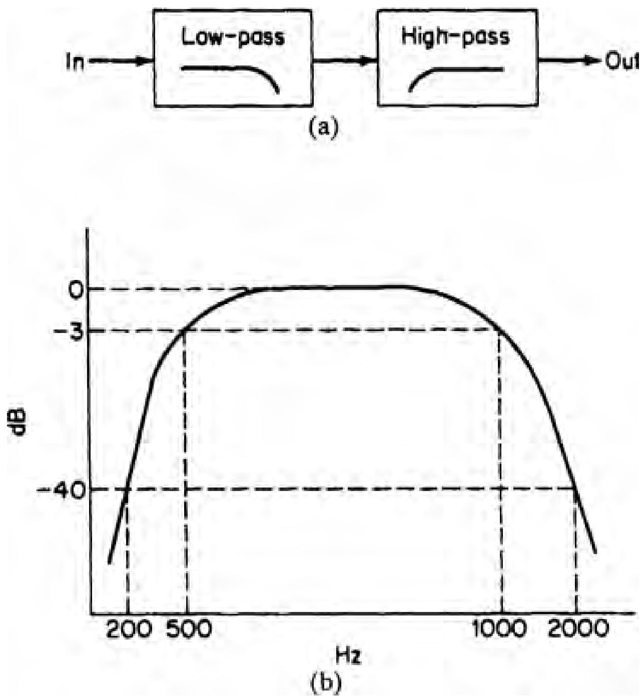

FIGURE 2-6 The results of Example 2-5: *(a)* cascade of low-pass and high-pass filters; and *(b)* frequency response.

In order to design a bandpass filter, the following sequence of steps is involved.

1. Convert the given bandpass filter requirement into a normalized low-pass specification.
2. Select a satisfactory low-pass filter from the normalized frequency-response curves.
3. Transform the normalized low-pass parameters into the required bandpass filter.

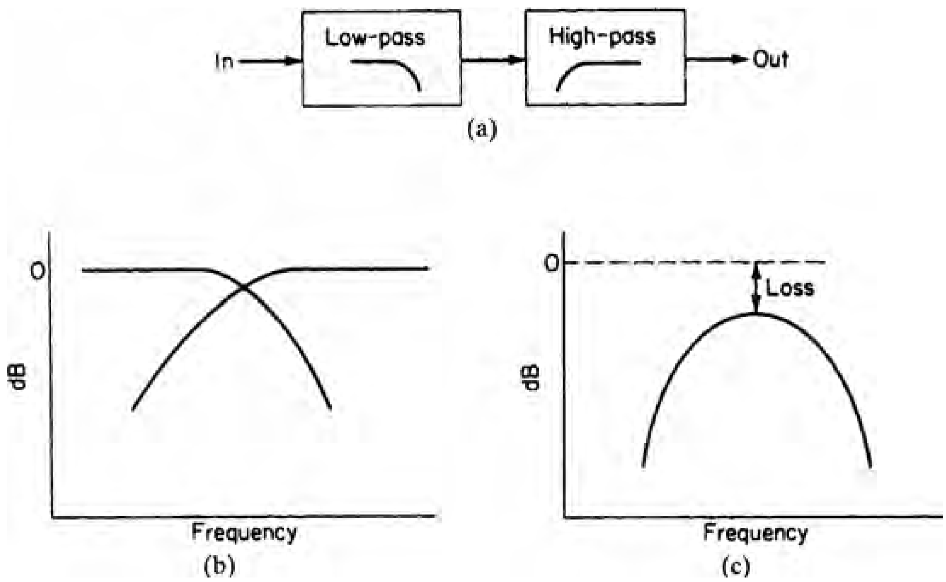

FIGURE 2-7 Limitations of the wideband approach for narrowband filters: (*a*) a cascade of low-pass and high-pass filters; (*b*) a composite response; and (*c*) algebraic sum of attenuation.

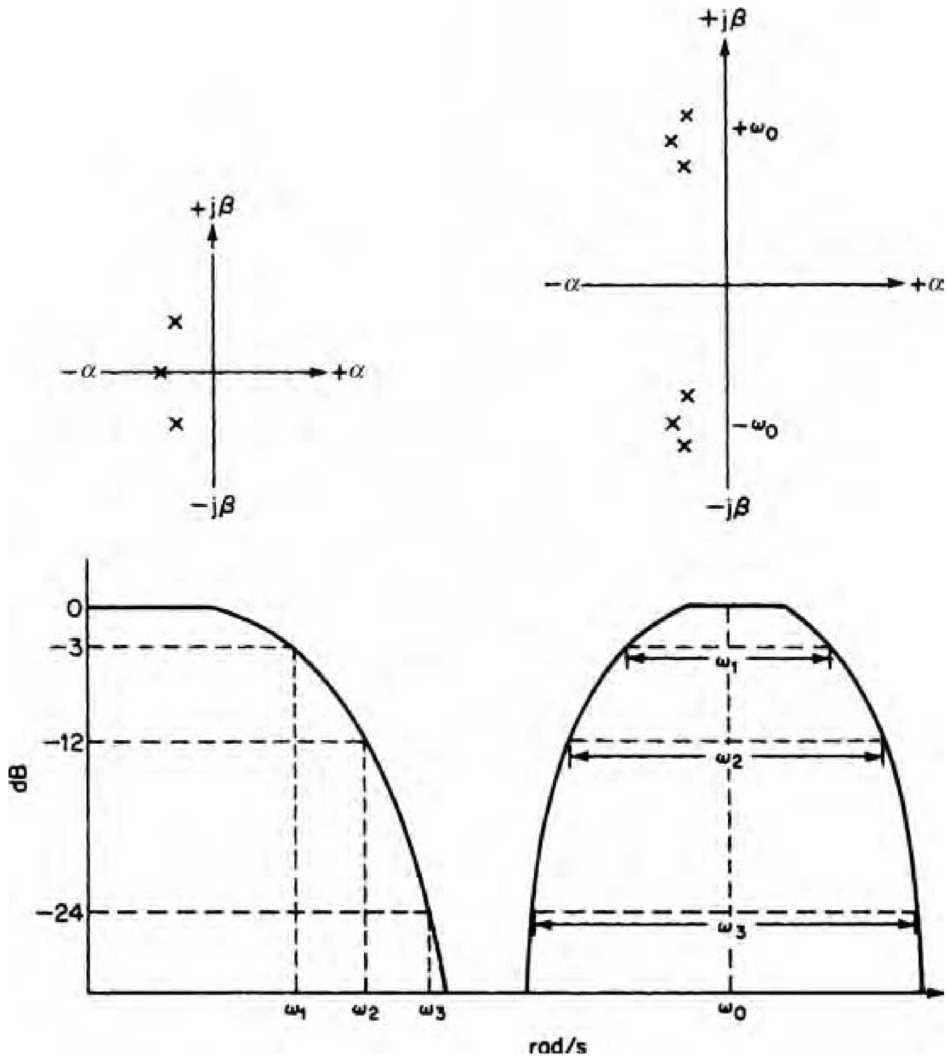

FIGURE 2-8 A low-pass to bandpass transformation.

The response shape of a bandpass filter is shown in Figure 2-9, along with some basic terminology. The center frequency is defined as

$$f_0 = \sqrt{f_L f_u} \tag{2-14}$$

where f_L is the lower passband limit and f_u is the upper passband limit, usually the 3-dB attenuation frequencies. For the more general case

$$f_0 = \sqrt{f_1 f_2} \tag{2-15}$$

where f_1 and f_2 are any two frequencies having equal attenuation. These relationships imply geometric symmetry; that is, the entire curve below f_0 is the mirror image of the curve above f_0 when plotted on a *logarithmic* frequency axis.

An important parameter of bandpass filters is the filter selectivity factor or Q, which is defined as

$$Q = \frac{f_0}{\mathrm{BW}} \tag{2-16}$$

where BW is the passband bandwidth or $f_u - f_L$.

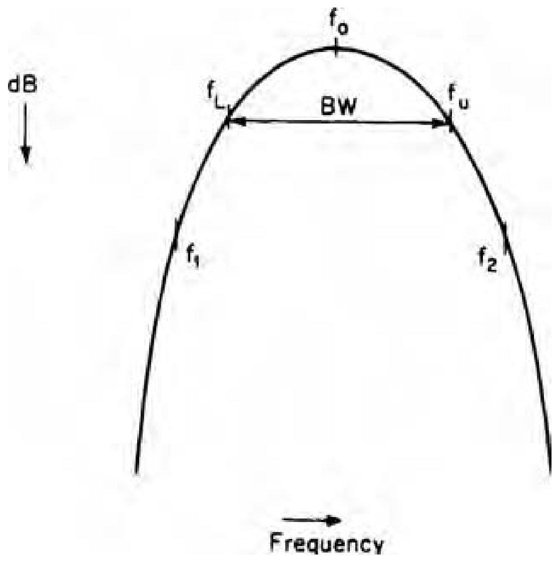

FIGURE 2-9 A general bandpass filter response shape.

As the filter Q increases, the response shape near the passband approaches the arithmetically symmetrical condition which is mirror-image symmetry near the center frequency, when plotted using a *linear* frequency axis. For Qs of 10 or more, the center frequency can be redefined as the arithmetic mean of the passband limits, so we can replace Equation (2-14) with

$$f_0 = \frac{f_L + f_u}{2} \tag{2-17}$$

In order to utilize the normalized low-pass filter frequency-response curves, a given narrowband bandpass filter specification must be transformed into a normalized low-pass requirement. This is accomplished by first manipulating the specification to make it geometrically symmetrical. At equivalent attenuation points, corresponding frequencies above and below f_0 must satisfy

$$f_1 f_2 = f_0^2 \tag{2-18}$$

which is an alternate form of Equation (2-15) for geometric symmetry. The given specification is modified by calculating the corresponding opposite geometric frequency for each stopband frequency specified. Each pair of stopband frequencies will result in two new frequency pairs. The pair having the lesser separation is retained, since it represents the more severe requirement.

A bandpass filter steepness factor can now be defined as

$$A_s = \frac{\text{stopband bandwidth}}{\text{passband bandwidth}} \tag{2-19}$$

This steepness factor is used to select a normalized low-pass filter from the frequency-response curves that makes the passband to stopband transition within a frequency ratio of A_s.

The following example shows the normalization of a bandpass filter requirement.

Example 2-6 Normalizing a Bandpass Filter Requirement

Required:
Normalize the following bandpass filter requirement:

A bandpass filter
A center frequency of 100 Hz
3 dB at ±15 Hz (85 Hz, 115 Hz)
40 dB at ±30 Hz (70 Hz, 130 Hz)

Result:

(*a*) First, compute the center frequency f_0.

$$f_0 = \sqrt{f_L f_u} = \sqrt{85 \times 115} = 98.9\ \text{Hz} \tag{2-14}$$

(*b*) Compute two geometrically related stopband frequency pairs for each pair of stopband frequencies given.

Let $f_1 = 70$ Hz.

$$f_2 = \frac{f_0^2}{f_1} = \frac{(98.9)^2}{70} = 139.7\ \text{Hz} \tag{2-18}$$

Let $f_2 = 130$ Hz.

$$f_1 = \frac{f_0^2}{f_2} = \frac{(98.9)^2}{130} = 75.2\ \text{Hz} \tag{2-18}$$

The two pairs are

$$f_1 = 70\ \text{Hz}, f_2 = 139.7\ \text{Hz}\ (f_2 - f_1 = 69.7\ \text{Hz})$$

and

$$f_1 = 75.2\ \text{Hz}, f_2 = 130\ \text{Hz}\ (f_2 - f_1 = 54.8\ \text{Hz})$$

Retain the second frequency pair, since it has the lesser separation. Figure 2-10 compares the specified filter requirement and the geometrically symmetrical equivalent.

(*c*) Calculate A_s.

$$A_s = \frac{\text{stopband bandwidth}}{\text{passband bandwidth}} = \frac{54.8\ \text{Hz}}{30\ \text{Hz}} = 1.83 \tag{2-19}$$

(*d*) A normalized low-pass filter can now be selected from the normalized curves. Since the passband limit is the 3-dB point, the normalized filter is required to have over 40 dB of rejection at 1.83 rad/s or 1.83 times the 1-rad/s cutoff.

The results of Example 2-6 indicate that when frequencies are specified in an arithmetically symmetrical manner, the narrower stopband bandwidth can be directly computed by

$$\text{BW}_{\text{stopband}} = f_2 - \frac{f_0^2}{f_2} \tag{2-20}$$

The narrower stopband bandwidth corresponds to the more stringent value of A_s, the steepness factor.

It is sometimes desirable to compute two geometrically related frequencies that correspond to a given bandwidth. Upon being given the center frequency f_0 and the bandwidth BW, the lower and upper frequencies are respectively computed by

$$f_1 = \sqrt{\left(\frac{\mathrm{BW}}{2}\right.} \tag{2-21}$$

$$f_2 = \sqrt{\left(\frac{\mathrm{BW}}{2}\right)^2 + f_0^2} + \frac{\mathrm{BW}}{2} \tag{2-22}$$

Use of these formulas is illustrated in the following example.

Example 2-7 Determining Bandpass Filter Bandwidths at Equal Attenuation Points

Required:

For a bandpass filter having a center frequency of 10 kHz, determine the frequencies corresponding to bandwidths of 100 Hz, 500 Hz, and 2000 Hz.