## * حالات خاصة للدالة $f(x)$

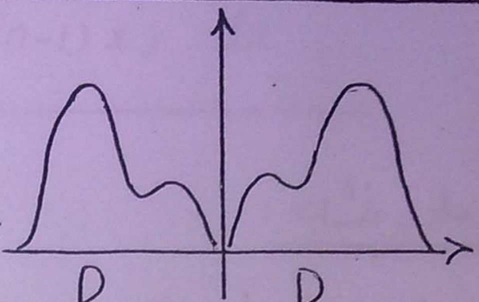
### 1) If $f(x)$ is an even function (دالة زوجية)

Called Cosine harmonic

$\Rightarrow b_n = \text{Zero}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$



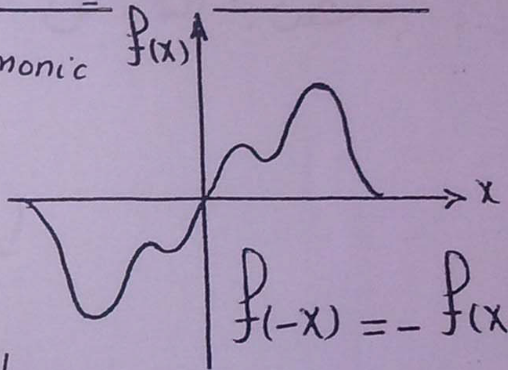
$f(-x) = f(x)$

### 2) If $f(x)$ is an odd function (دالة فردية)

Called Sine harmonic

$\Rightarrow a_0 = \text{Zero}$ & $a_n = \text{Zero}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$




$f(-x) = -f(x)$

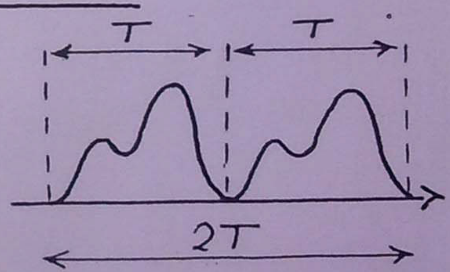
### 3) If $f(x+\pi) = f(x) \Rightarrow$ even harmonic

$a_{2n-1} = 0$ & $b_{2n-1} = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_{2n} = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(2nx)\,dx$$

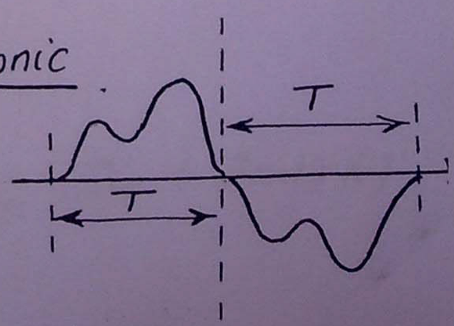
$$b_{2n} = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(2nx)\,dx$$




### 4) If $f(x+\pi) = -f(x) \Rightarrow$ odd harmonic

$a_0 = 0$ , $a_{2n} = 0$ , $b_{2n} = 0$

&



$$a_{2n-1} = \frac{2}{\pi} \int_0^{\pi} f(x) \cos((2n-1)x)\, dx$$

$$b_{2n-1} = \frac{2}{\pi} \int_0^{\pi} f(x) \sin((2n-1)x)\, dx$$

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**ملحوظة :**

لاحظ أن في جميع الحالات السابقة نستخدم نفس قوانين
$a_0$ , $a_n$ , $b_n$ ولكن نكامل على أي فترة طولها $\pi$
ونضرب التكامل في 2 .

## Fourier Series in Case of

### I) Cosine harmonic ( even function )

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx$$

### II) Sine harmonic ( odd function )

$$f(x) = \sum_{n=1} b_n \sin nx$$

### III) Even harmonic

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_{2n} \cos(2nx) + b_{2n} \sin(2nx)$$

### IV) Odd harmonic

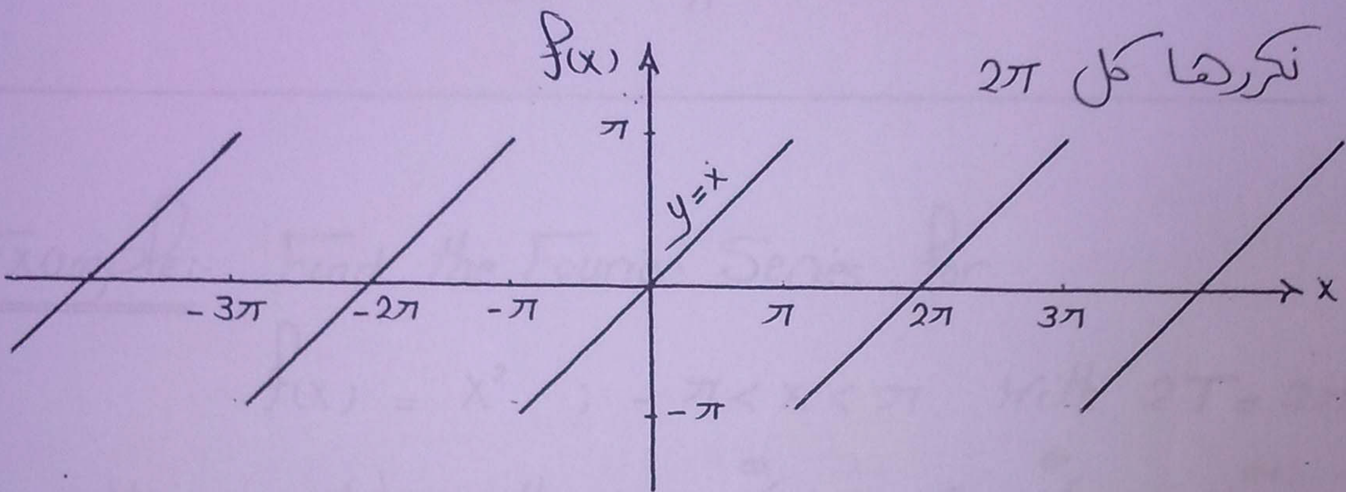
$$f(x) = \sum_{n=1} a_{2n-1} \cos((2n-1)x) + b_{2n-1} \sin((2n-1)x).$$

**Example:** Find the Fourier Series for

$$f(x) = x \quad ; \quad -\pi < x < \pi \quad \text{With } 2T = 2\pi$$

**Solution**

نرسم المنحنى $y = x$ فى الفتره من $x = -\pi$ الى $x = \pi$ ثم نكررها كل $2\pi$




We have $f(x)$ is an odd function ( داله فرديه )

$$\Rightarrow \quad a_0 = 0 \quad , \quad a_n = 0$$

$$\& \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx \quad \Rightarrow$$ نستخدم التكامل بالتجزئ

$$= \frac{2}{\pi} \left( x \left( \frac{-\cos nx}{n} \right) - \left( \frac{-\sin nx}{n^2} \right) \right) \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \left( \frac{-\pi \cos n\pi}{n} + \underbrace{\cancel{\frac{\sin n\pi}{n^2}}}_{\text{Zero}} + \text{Zero} - \underbrace{\cancel{\frac{\sin(\text{Zero})}{n^2}}}_{\text{Zero}} \right)$$

$$b_n = \frac{-2\cos n\pi}{n} = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Fourier Series $\quad f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx + b_n \sin nx$

$$\Rightarrow \quad f(x) = \sum_{n=1} \frac{2(-1)^{n+1}}{n} \sin nx$$

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<u>Example:-</u> Find the Fourier Series for

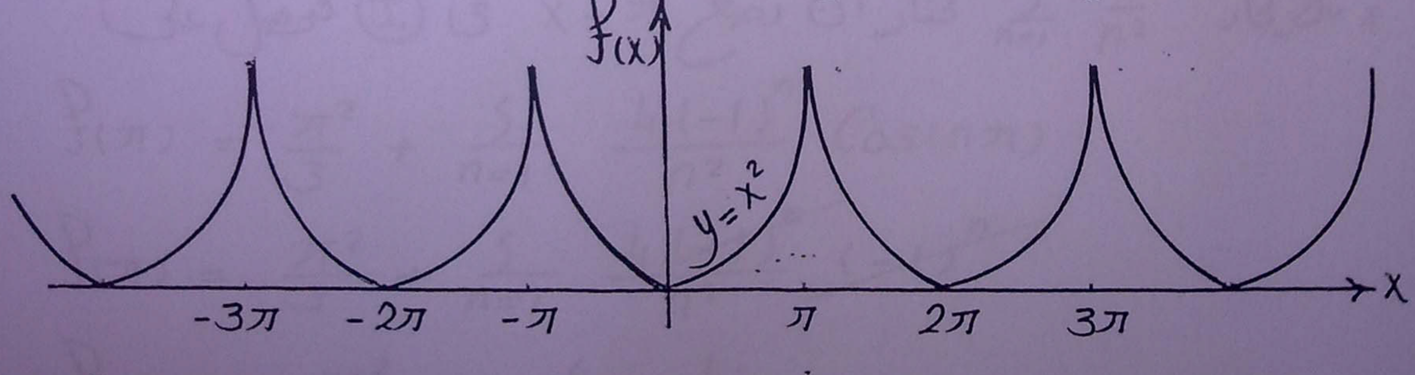
$$f(x) = x^2 \quad ; \; -\pi < x < \pi \quad \text{with } 2T = 2\pi$$

Hence, obtain the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

<u>Solution:-</u>

نرسم المنحنى $y = x^2$ فى الفتره من $x = -\pi$ الى

$x = \pi$ ثم نكررها كل $2\pi$




$f(x)$ is an even function ( دالـــه زوجيـــه )

$b_n = 0$ &

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\Big|_0^{\pi}\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

نستخدم التجزئ

$$= \frac{2}{\pi}\left(x^2\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right)\Big|_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left(2\pi\frac{\cos n\pi}{n^2}\right) = \frac{4\cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

<u>Fourier Series</u> $f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx + b_n \sin nx$

$$\Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4(-1)^n}{n^2}\cos nx \longrightarrow \text{(I)}$$

* لإيجاد $\sum_{n=1}^{\infty} \frac{1}{n^2}$ نختار أن نضع $x = \pi$ في (I) فنحصل على

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4(-1)^n}{n^2}\cos(n\pi)$$

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4\cancel{(-1)^n}}{n^2}\cancel{(-1)^n}$$

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4}{n^2}$$

$\hookrightarrow$ من منحنى $f(x)$ نجد أن قيمتها هي $\pi^2$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1} \frac{4}{n^2}$$

$$\Rightarrow 4 \sum_{n=1} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

* لإيجاد $\sum_{n=1} \frac{(-1)^{n+1}}{n^2}$ نختار أن نعوض بـ $x = 0$ في (I)

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1} \frac{4(-1)^n}{n^2} \underbrace{\cos(\text{Zero})}_{1}$$

نجد من منحنى $f(x)$ أن قيمتها عند $x = 0$ هي Zero

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1} \frac{(-1)^n}{n^2}$$

$$0 = \frac{\pi^2}{3} - 4 \sum_{n=1} \frac{(-1)^{n+1}}{n^2}$$

$$\Rightarrow \sum_{n=1} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

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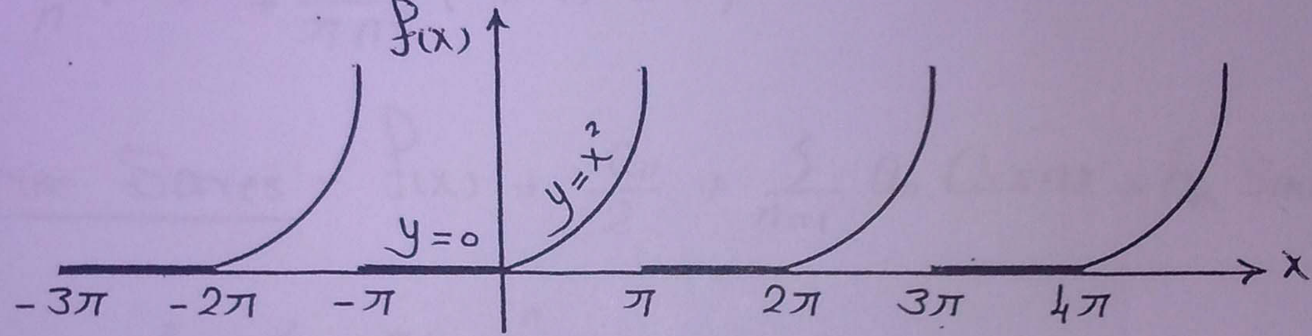
__Example :-__ Find Fourier Series for

$$f(x) = \begin{cases} 0 & ; -\pi < x < 0 \\ x^2 & ; 0 < x < \pi \end{cases} \qquad \text{With period } 2\pi$$

__Solution :-__

* في الفترة من $x = -\pi$ الى $x = 0$ الدالة $f(x) =$ صفر فترسم

على محور $x$ , وفي الفترة $x = 0$ الى $x = \pi$ تأخذ شكل $y = x^2$




$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2\,dx\right)$$

$$\Rightarrow \quad a_0 = \frac{1}{\pi}\int_{0}^{\pi} x^2\,dx = \frac{1}{\pi}\left(\frac{x^3}{3}\Big|_0^{\pi}\right) = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left(x^2\,\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2}\right) = \frac{2(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left( x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left( \frac{-\pi^2 \cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2}{n^3}\right)$$

$$= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)$$

<u>Fourier Series</u> : $f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos nx + b_n \sin n$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1} \frac{2(-1)^n}{n^2}\cos nx$$
$$+ \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right)\sin nx .$$

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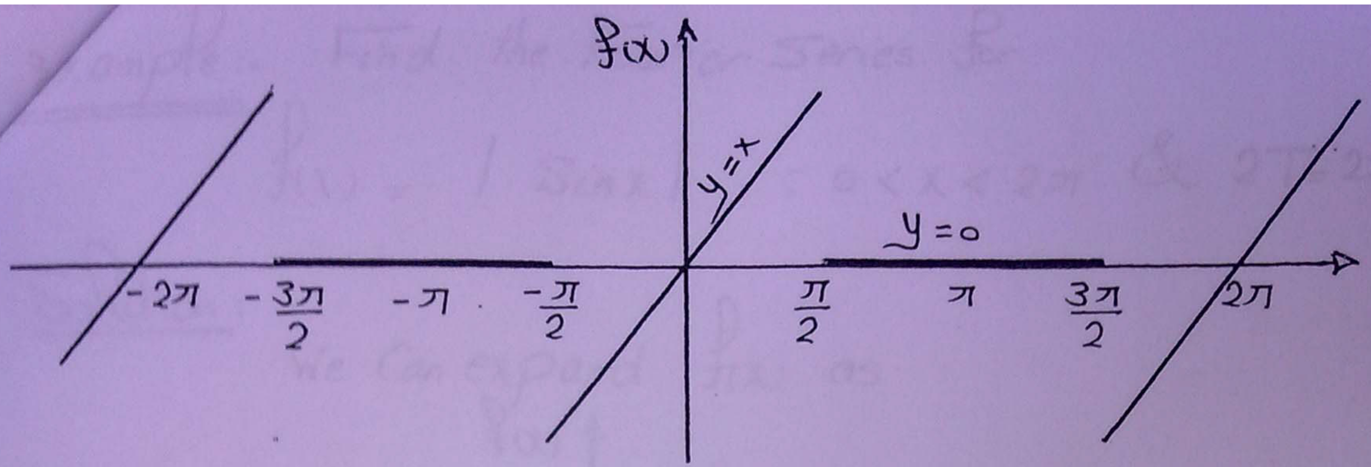
**<u>Example</u>** :- Find Fourier Series for

$$f(x) = \begin{cases} x & ; \; -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & ; \; \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases} \qquad \text{with } 2T = 2\pi$$

<u>Solution</u> :- نرسم $y = x$ في الفترة من $x = -\frac{\pi}{2}$ الى $x = \frac{\pi}{2}$

ثم نرسم $y = 0$ (محور $x$) في الفترة من $x = \frac{\pi}{2}$ الى $x = \frac{3\pi}{2}$




$f(x)$ is an odd function ( داله فردية )

$\Rightarrow \quad a_0 = 0 \quad , \quad a_n = 0 \quad \&$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \sin nx \, dx$$ $\Rightarrow$ أو نكامل على أى فترة طولها $2\pi$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\frac{3\pi}{2}} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left( \int_{-\pi/2}^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{3\pi/2} \text{zero} \sin nx \, dx \right)$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x \sin nx \, dx$$ $\longrightarrow$ نستخدم التكامل بالتجزئ

$$= \frac{1}{\pi}\left( x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\Bigg( -\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right)$$
$$+ \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) \Bigg)$$

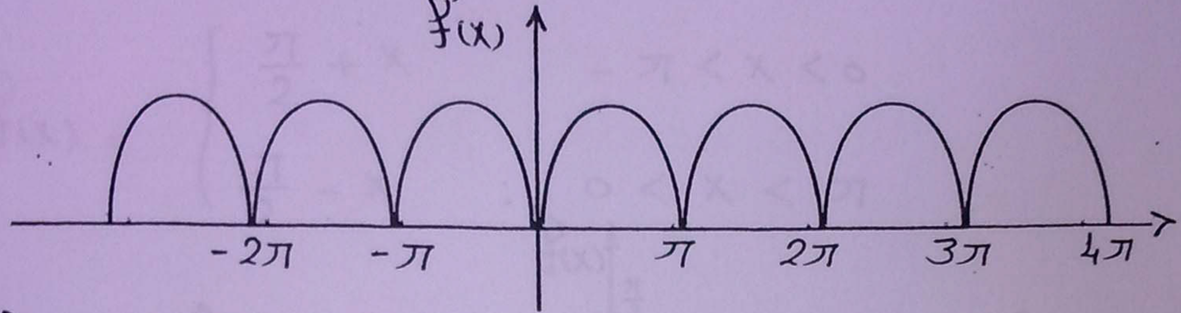
$$f(x) = \sum_{n=1}\left( \frac{-\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2}\right) \cdot \sin nx$$

**Example:-** Find the Fourier Series for

$$f(x) = |\sin x| \quad ; \quad 0 < x < 2\pi \quad \& \quad 2T = 2\pi$$

**Solution:**

We Can expand $f(x)$ as




$$f(x) = \begin{cases} \sin x & ; \quad 0 < x < \pi \\ -\sin x & ; \quad \pi < x < 2\pi \end{cases}$$

$f(x)$ is an even function (دالـــة زوجيـه) & has even harmonic $( f(x+\pi) = f(x) ) \Rightarrow b_n = 0$ ,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \, dx = \frac{2}{\pi} \left( -\cos x \Big|_0^{\pi} \right) = \frac{4}{\pi}$$

$$a_{2n} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos 2nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos 2nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(1+2n)x + \sin(1-2n)x \quad dx$$

$$= \frac{1}{\pi} \left( -\frac{\cos(1+2n)x}{1+2n} - \frac{\cos(1-2n)x}{1-2n} \right) \Big|_0^{\pi} = \frac{2}{\pi} \left( \frac{1}{2n+1} + \frac{1}{1-2n} \right)$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1} \frac{2}{\pi} \left( \frac{1}{2n+1} + \frac{1}{1-2n} \right) \cos 2nx \, .$$

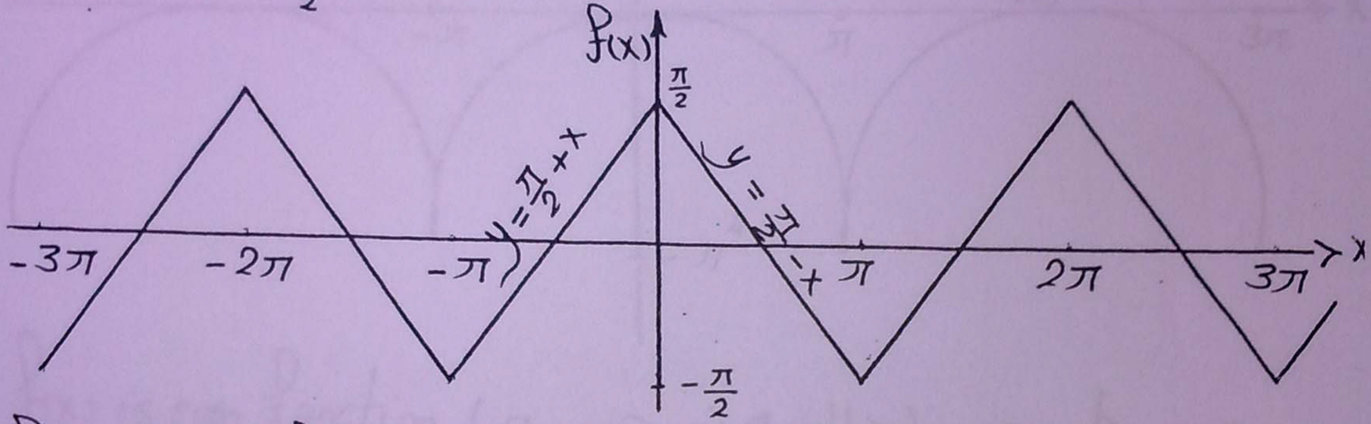
**Example:** Find the Fourier Series for

$$f(x) = \frac{\pi}{2} - |x| \quad ; \quad -\pi < x < \pi \quad \& \quad 2T = 2\pi$$

**Solution:**

We expand $f(x)$ as $\left( \text{Since } |x| = \begin{cases} x & \text{for } x > 0 \\ -x & \text{for } x < 0 \end{cases} \right.$

$$f(x) = \begin{cases} \frac{\pi}{2} + x & ; \ -\pi < x < 0 \\ \frac{\pi}{2} - x & ; \ 0 < x < \pi \end{cases}$$




$f(x)$ is even function ( دالــه زوجيـه ) & has odd harmonic $\left( f(x+\pi) = -f(x) \right) \Rightarrow b_n = 0 \ , \ a_0 = 0$

$$\& \quad a_{2n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(2n-1)x \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(2n-1)x \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \left( \frac{\pi}{2} - x \right) \cos(2n-1)x \, dx$$

$$= \frac{2}{\pi} \left( \left( \frac{\pi}{2} - x \right) \frac{\sin(2n-1)x}{2n-1} - (-1) \left( \frac{-\cos(2n-1)x}{(2n-1)^2} \right) \right) \Bigg|_{0}^{\pi}$$

$$= 2/\pi \left( 2/(2n-1)^2 \right)$$

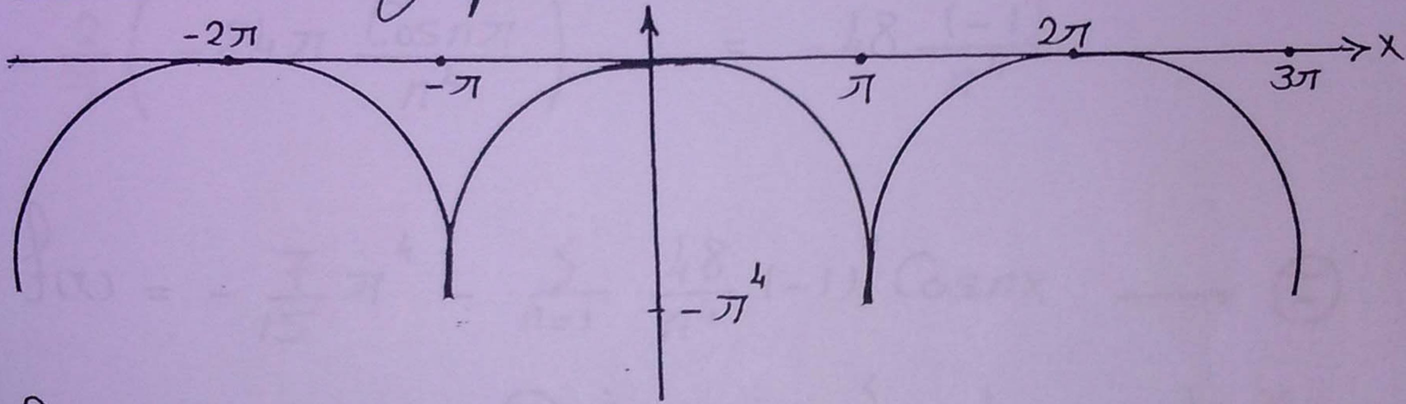
$$f(x) = \sum_{n=1} \frac{4}{\pi (2n-1)^2} \cos(2n-1)x .$$

**xample:-** Find the Fourier Series for

$$f(x) = x^4 - 2\pi^2 x^2 \quad ; \quad -\pi < x < \pi \quad \& \quad 2T = 2\pi$$

Hence, deduce the sum $1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

**Solution:-** The graph of $f(x)$ is as follow:-




$f(x)$ is even function ( دالــة زوجيــة ) $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^4 - 2\pi^2 x^2 \, dx$$

$$= \frac{2}{\pi}\left(\frac{x^5}{5} - \frac{2\pi^2}{3}x^3\right)\Big|_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^5}{5} - \frac{2\pi^5}{3}\right) = -\frac{14}{15}\pi^4$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x^4 - 2\pi^2 x^2)\,\cos nx \, dx.$$

نستخدم التكامل بالتجزئ

$$= \frac{2}{\pi}\Bigg( (x^4 - 2\pi^2 x^2)\frac{\sin nx}{n} - (4x^3 - 4\pi^2 x)\left(\frac{-\cos nx}{n^2}\right)$$

$$+ (12x^2 - 4\pi^2)\left(\frac{-\sin nx}{n^3}\right) - (24x)\left(\frac{\cos nx}{n^4}\right)$$

$$+ 24\left(\frac{\sin nx}{n^5}\right)\Bigg)\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left(-24\pi\frac{\cos n\pi}{n^4}\right) = -48\frac{(-1)^n}{n^4}$$

$$f(x) = -\frac{7}{15}\pi^4 - \sum_{n=1} \frac{48}{n^4}(-1)^n \cos nx \longrightarrow \text{(I)}$$

* لإيجاد $\sum_{n=1} \frac{1}{n^4}$ نعوض في (I) عن $x$ ب $\pi$

$$f(\pi) = -\frac{7}{15}\pi^4 - \sum_{n=1} \frac{48}{n^4}\cancel{(-1)^n}\,\cancel{(-1)^n}$$

$\downarrow$

نوجد قيمة الدالة من المعطى عند $x=\pi$ فتكون $-\pi^4$

$$-\pi^4 = -\frac{7}{15}\pi^4 - 48 \cdot \sum_{n=1} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1} \frac{1}{n^4} = \frac{1}{48}\left(\pi^4 - \frac{7}{15}\pi^4\right)$$

$$= \frac{\pi^4}{90}$$